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Greedy primal-dual algorithm for dynamic resource allocation in complex networks

Alexander L. Stolyar

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Abstract In Stolyar (Queueing Systems 50 (2005) 401–457) a dynamic control strategy, called greedy primal-dual (GPD) algorithm, was introduced for the problem of maximizing queueing network utility subject to stability of the queues, and was proved to be (asymptotically) optimal. (The network utility is a concave function of the average rates at which the network generates several "commodities.") Underlying the control problem of Stolyar (Queueing Systems 50 (2005) 401–457) is a convex optimization problem subject to a set of *linear* constraints.

In this paper we introduce a generalized GPD algorithm, which applies to the network control problem with additional *convex* (possibly non-linear) constraints on the average commodity rates. The underlying optimization problem in this case is a convex problem subject to convex constraints. We prove asymptotic optimality of the generalized GPD algorithm. We illustrate key features and applications of the algorithm on simple examples.

Keywords Queueing networks · Dynamic scheduling · Resource allocation · Convex optimization · Non-linear constraints · Greedy primal-dual algorithm

AMS Subject Classifications: 90B15 · 90C25 · 60K25 · 68M12

1 Introduction

This paper is a natural progression of [15], where a dynamic control strategy, called greedy primal-dual (GPD) algorithm, was introduced for the problem of maximizing queueing network utility subject to stability of the queues, and was proved to be (asymptotically) optimal. This problem accommodates a large variety of communication network applications, including the utility based network congestion control [9, 11] and many resource allocation problems in wireless systems. (See [15] for a review of the model applications.) Underlying the network control problem addressed in [15] is a convex optimization problem subject to a set of *linear* constraints. However, as we will illustrate in Sections 3 and 7, some network control problems arising in applications are such that the underlying optimization problems are convex, but with a set of *convex* (not necessarily linear) constraints. In this paper we introduce a generalization of GPD control algorithm and prove its asymptotic optimality for such more general problems.

More specifically, the model in [15] is such that each control action has associated impact on the network queues (namely, it determines the rates of exogenous arrivals, service rates, and routing of served customers between the queues), and also generates certain amounts of several *commodities* (which can be amounts of traffic, "costs," etc.). The utility of the network is a concave function of the vector x of average commodity generation rates. (Utility function need *not* be strictly concave.) As demonstrated in [15], the convex optimization problem that underlies the network control problem of maximizing utility subject to queueing stability, has the following form:

$$\max_{x \in V} H(x) \tag{1}$$

subject to

$$G_j(x) \le 0, \quad j \in \mathcal{J}, \tag{2}$$

A. L. Stolyar (✉)
Bell Labs, Lucent Technologies, 600 Mountain Ave., 2C-322,
Murray Hill, NJ 07974
e-mail: stolyar@research.bell-labs.com

where $V \subset R^N$ is a compact convex *system rate region*, H is a concave utility function, and all constraints (2) are linear, corresponding to the queueing stability requirement—one constraint per queue. The GPD algorithm of [15] is asymptotically optimal in that, under this algorithm, the commodity rates converge to an optimal solution of the underlying problem (1)–(2). The algorithm is of *primal-dual* type, so that, roughly speaking, commodity rates are primal and (rescaled) queue lengths are dual variables for the problem (1)–(2). (In an independent parallel effort to [15], papers [7, 10, 12] propose algorithms for related network control problems. The algorithms of [7, 10, 12] are of dual type and, unlike the GPD algorithm, they additionally require that utility function is separable with respect to individual commodity rates x_n, i.e. $H(x) = \sum_n H_n(x_n)$, with each function $H_n(\cdot)$ being strictly concave and increasing.)

Suppose now that the network control problem is as above, but there are additional *convex* constraints on the average commodity rates. Then, the underlying optimization problem still has the form (1)–(2), but the set of constraints (2) is extended to include additional ones with G_j being just convex, possibly non-linear. In Section 6 of this paper we introduce a generalized GPD algorithm, that is applicable to such more general problems, and prove its asymptotic optimality. For each additional (convex) constraint in (2) the algorithm creates and "maintains" a *virtual queue length,* which (after appropriate rescaling) converges to an optimal dual variable (Lagrange multiplier) for the constraint. The specific *update rule for the virtual queues corresponding to convex constraints*, is the key new part of the generalized GPD algorithm, compared to that of [15]. Just as the rest of the algorithm, this update rule (see (51)) is very "parsimonious"—it only requires the knowledge of the values and gradients of the constraint functions at a single point given by the current (estimated) average commodity rates.

The meaning of the GPD algorithm asymptotic optimality, as will be explained in more detail in Section 6.5, is roughly as follows. Given a fixed (small) parameter $\beta > 0$ of the algorithm, within a time interval of the order of $1/\beta$ the algorithm brings the system from any initial state to a *regime* in which average commodity rates (over $1/\beta$ long time intervals) are (close to) optimal. Depending on the time-scale relevant to a specific application, those average rates may be long- or short-term averages, or even be considered as "instantaneous" rates.

At this point, we would like to emphasize that standard and well studied algorithms for convex optimization (cf. [3, 5]), including, for example, the classical Arrow-Hurwicz-Uzawa primal-dual algorithm [2], are often inapplicable to the *dynamic control of queueing networks* problems we address in this paper, as well as in [15]. (This will be discussed in detail in Section 3.1, using the application example described there.)

As in [15], the key part of the proof of GPD algorithm asymptotic optimality is the analysis of the dynamic system whose trajectories (called GPD-trajectories) arise as asymptotic limits of the network evolution under the GPD algorithm. The main result of this paper (Theorem 1) shows that the GPD-trajectories are attracted to the optimal primal-dual solution pairs for the problem (1)–(2).

The summary of the main contributions of this paper is as follows.

- We extend the GPD algorithm of [15] and the proof of its asymptotic optimality to a much broader queueing network control problem, allowing general convex constraints on the average commodity rates. From the technical point of view, the key contributions are the introduction of the new update rule for the virtual queue lengths (corresponding to convex constraints) and proof of the attraction property of the resulting (more general) GPD-trajectories.
- Using examples, we demonstrate that additional convex constraints may indeed arise in applications, and show that in many cases the general GPD algorithm specializes to a rather simple distributed dynamic network control strategy.
- We believe that our main results contribute to the convex optimization theory as well, since the GPD algorithm can naturally be viewed as a dynamic mechanism for solving rather general convex optimization problems. Our main focus here is on devising and analyzing an algorithm that can be used for dynamic control of queueing networks in cases where standard convex optimization techniques cannot be applied. (An interesting question is to compare the efficiency of the GPD algorithm to that of standard convex optimization methods, when both are applied to "conventional" convex optimization problems. This question, however, is outside the scope of this paper, and may be a subject of future research.)

The rest of the paper is organized as follows. Section 2 introduces some basic notation. In Section 3, to motivate our general model, we describes two examples of network control problems that involve non-linear convex constraints. In Section 4 we formally define GPD-trajectories associated with problems of type (1)–(2), and formulate our main result, Theorem 1. Section 5 is the proof of Theorem 1. In Section 6 we introduce our general resource allocation model, define the GPD algorithm, and prove its asymptotic optimality. (To simplify exposition, in most of Section 6 we restrict ourselves to a *model* that is a special case of that of [15], but consider a more general *problem* for it. The general form of GPD algorithm, that applies to the queueing network model of [15] and the more general problem, is presented in Section 6.6.) In Section 7 we return to the examples of Section 3, to show solutions provided by the GPD algorithm and illustrate its key features. We should note that Sections 4–5 and Section 6 are virtually independent, and can be read in any order. (It may help if reader skims through Section 4 before reading Section 6.) We also note that although all the model and algorithm definitions and result formulations do not require familiarity with paper [15], virtually all *proofs* heavily rely on those in [15].

2 Basic notation

We denote by R, R_+, and R_-, the sets of real, real non-negative and real non-positive numbers, respectively. Corresponding N-times product spaces are denoted R^N, R_+^N, and R_-^N. The space R^N is viewed as a standard vector-space, with elements $x \in R^N$ being row-vectors $x = (x_1, \ldots, x_N)$. The scalar product of $x, y \in R^N$ is

$$x \cdot y \doteq \sum_{n=1}^{N} x_n y_n \ ;$$

and the norm of x is

$$\|x\| \doteq \sqrt{x \cdot x} \ .$$

We denote by

$$\rho(x, y) \doteq \|x - y\|$$

the *distance* between vectors x and y in R^N, and by

$$\rho(x, V) \doteq \inf_{y \in V} \rho(x, y)$$

the distance between a vector $x \in R^N$ and a set $V \subseteq R^N$. If $(x(t),\ t \geq 0)$ and V is a vector function and a set, respectively, in R^N, the convergence $x(t) \to V$ means that $\rho(x(t), V) \to 0$ as $t \to \infty$.

For a set V and a scalar function $W(v),\ v \in V$,

$$\arg\max_{v \in V} W(v)$$

denotes the *subset* of vectors $v \in V$ which maximize $W(v)$.

For $\xi, \eta \in R$, we denote $\xi \wedge \eta \doteq \min\{\xi, \eta\}$, $\xi \vee \eta \doteq \max\{\xi, \eta\}$, $\xi^+ \doteq \max\{\xi, 0\}$; for $\xi \in R$ and $\eta \in R_+$, $[\xi]_\eta^+ = \xi$ if $\eta > 0$, and $[\xi]_\eta^+ = \xi^+$ if $\eta = 0$.

Abbreviation *u.o.c.* means *uniform on compact sets* convergence of functions. The term *almost everywhere (a.e.)* means almost everywhere with respect to Lebesgue measure.

We denote by $D_{R^N}[0, \infty)$ the Skorohod space of functions with domain $[0, \infty)$, taking values in R^N, $N \geq 1$, which are right-continuous and have left-limits. The subspace of $D_{R^N}[0, \infty)$ consisting of continuous functions is denoted by $C_{R^N}[0, \infty)$; notation $C_{R_+^N}[0, \infty)$ is used for the subset of $C_{R^N}[0, \infty)$ consisting of functions with values in R_+^N. (Topologies, σ-algebras, and norms on these spaces are specified later, where and when necessary.)

3 Motivating examples

In this section we describe two examples of dynamic network control problems, that have non-linear convex constraints, which can be accommodated by the extended GPD algorithm, introduced and proved to be asymptotically optimal in Section 6. (These constraints *cannot* be handled by the GPD algorithm of [15].) We formally introduce and discuss the models and the corresponding problems. We will return to these examples later in Section 7, where we demonstrate that the extended GPD algorithm can be applied to both problems to produce (asymptotically) optimal dynamic control algorithms.

The example of Section 3.1 is rather simple. Its purpose is to, first, remind (following [15]) why standard methods of convex optimization are not applicable to many problems of queueing network control and, second, illustrate that additional convex constraints can naturally arise in such problems. The example of Section 3.2 describes convex constraints of a different type; it will also allow us to illustrate (in Section 7) that the GPD algorithm often allows a "distributed implementation" in a network.

3.1 Example 1

We consider a model where a wireless network "base station" (or an "access point", or simply a network node) sends data flows to several wireless (mobile) data users. The basic problem is to dynamically schedule data transmissions to the users so as to minimize average power consumption subject to the constraint that the aggregate traffic flow "utility" is above certain minimum level.

An important feature of the model is that the users share radio channel, with the base station dynamically allocating data transmission rates and transmission powers to the users. Moreover, the channel capacity is asynchronously time-varying with respect to different users. In other words, roughly speaking and assuming for simplicity that the transmission power is fixed, at one point in time the available transmission rate may be high for one user and low for another, in which case there is an incentive to "opportunistically" allocate a larger fraction of channel time to the former user transmission; and at a different time point the situation may be reversed. Thus, base station can utilize *opportunistic* *scheduling* (cf. [1, 16]) to improve overall efficiency of the radio channel.

Formally, the model is as follows. A base station sends data traffic to a finite set $\mathcal{N} = \{1, \dots, N\}$ of wireless users, indexed by n. We assume that the time is slotted, indexed by $t = 0, 1, 2, \dots,$ and use slot duration as the time unit. The state $m(t)$ of the wireless channel follows an irreducible Markov chain with the finite state space M. When the channel state is $m \in M$, the scheduling decisions k available to the base station form a finite set $K(m)$. If the base station chooses decision $k \in K(m)$ in time slot t, then in this slot it sends the amounts of data (say, the number of bytes) $b_n(k) \geq 0$ to the users $n \in \mathcal{N}$, and this action consumes the amount $w(k)$ of energy. (Note that, in principle, a scheduling decision may be such that several users transmit data simultaneously. The situation where only one user is allowed to transmit at a time is a special case.)

Let us denote by x_n the average rate of flow n (i.e., the average value of b_n over time), and by x_0 the average power usage (that is the average value of w). Finally, the "utility" of flow n is $H_n(x_n)$, where $H_n(z)$, $z \geq 0$, is a continuously differentiable concave function. (For example, a common choice of a utility function is $H_n(z) = \log z$ - in fact, it is used by the standard scheduling algorithms in some commercial wireless technologies [4]. See [8, 9] for a general rationale for using concave utility functions of the traffic rates.) The problem is to find a dynamic scheduling strategy which minimizes the average power consumption, subject to some lower bound on the aggregate utility. Namely, the problem is to

$$\text{minimize } x_0 \tag{3}$$

subject to

$$\sum_{n\in\mathcal{N}} H_n(x_n) \geq h^{\min}, \tag{4}$$

where $h^{\min}$ is a given constant.

The convex optimization problem *underlying* control problem (3)–(4) is as follows:

$$\max_{x=(x_0,x_1,\dots,x_N)\in V} -x_0 \tag{5}$$

subject to

$$-\sum_{n\in\mathcal{N}} H_n(x_n) + h^{\min} \leq 0, \tag{6}$$

where $V \subset R^{N+1}$ is the region of all possible vectors of average rates $x = (x_0, x_1, \dots, x_N)$ under all scheduling strategies. The region V is a convex compact set, as explained later in Section 6 (and in the previous work, cf. [13, 15]).

Although the underlying problem (5)–(6) looks fairly standard, the "standard" convex optimization algorithms (cf. [2, 3, 5]) *cannot* be applied to devise a control strategy solving problem (3)–(4). One difficulty is that region V is *not given explicitly*. In our case, for example, it depends on the stationary distribution of the channel state process $m(t)$ and on the sets of available scheduling decisions in each state. In typical wireless applications, none of this information is known in advance and in fact may change over time. This essentially precludes the use of scheduling algorithms utilizing some kind of advance "off-line" optimization. Standard dynamic primal-dual algorithms (cf. [2, 5]) can not be applied either, because—roughly speaking—they require that at every step primal variables (average rates x_n in our case) and dual variables are changed in certain directions. Our model however is such that in each time slot the base station has to pick one of the scheduling decisions (with corresponding b_n's and w) out of the finite number of instantaneously available choices. A scheduling strategy which would attempt to produce the desired short-term average traffic rates (so that x_n's are moved in the desired direction) would again require a priori knowledge of the stationary distribution of the channel state process, and thus would be infeasible to implement.

This difficulty (of V not being known explicitly) can be overcome by the GPD algorithm of [15] as long as the additional explicit constraints (4) are linear, by introducing virtual queues and mapping such constraints into the stability requirements of the queues. (For example, Section 5.2 of [15] gives an asymptotically optimal algorithm for maximizing aggregate utility $\sum_n H_n(x_n)$ subject to the average power usage constraint $x_0 \leq w^{\max}$). However, the constraint (4) on the minimum utility is typically non-linear—that is why for the problem (3)–(4) we need to apply the extended GPD algorithm of this paper. An asymptotically optimal algorithm for (3)–(4) will be given in Section 7.1.

Remark. The extended GPD algorithm (as defined in Section 6.6) applies to the following more general model as well. We can assume that the traffic sent to users $n \in \mathcal{N}$ has to further go through ("be processed by") a complex (time-varying) network of (interdependent) nodes. The "processing network" is then a *queueing network* of the type studied in [15] and described in Section 6.6 of this paper. The more general problem is to minimize average power usage by the base station (or several base stations) subject to the constraint on the utility of the flows and the constraint that network queues remain stable. The extension of the algorithm given in Section 7.1 is straightforward. We do not consider the more general problem here in order to focus attention on the additional convex constraints in the simplest possible scenario.

3.2 Example 2

The second example is a system where multiple wireless data users access a wireline communication network via access points. Each user sends data flow that starts at one of the access points, follows a fixed route through the wireline network, and terminates at one of its nodes. The data rates of the flows are not given in advance, but rather allocated to the flows by the network. In particular, each access point controls transmissions from "its" users and can utilize opportunistic scheduling (as discussed in Section 3.1). The goal of the network is to maximize system utility, which is a concave function of the flows' average rates, under the constraint that a "congestion cost" (say, average delay) along each flow route stays below a predefined bound. As we will see, these route congestion cost constraints are typically convex nonlinear, and thus solving the problem requires the extended GPD algorithm of Section 6.

The formal model is as follows. There is a finite set $\mathcal{N} = \{1, \dots, N\}$ of traffic sources, or wireless users, indexed by n. There is a finite set $\mathcal{I}$ of wireless access points i, via which users access a wireline network. (The opportunistic scheduling model that governs how users transmit data to access points will be described shortly.) The wireline network consists of a finite set $\mathcal{L}$ of communication links, indexed by ℓ. Each user generates a traffic flow that goes through one of the access points $i \in \mathcal{I}$, follows a fixed route through the wireline network, and terminates at one of its nodes; we use notation $\mathcal{R}(n)$ for the set of links ℓ traversed by flow n, and notation $\mathcal{R}^{-1}(\ell)$ for the set of flows traversing link ℓ.

Let us denote by x_n the average rate of flow n, and by $x^{(\ell)} \doteq \sum_{n\in\mathcal{R}^{-1}(\ell)} x_n$ the average rate of the total data flow traversing link ℓ.

Each link has corresponding "congestion cost" $C_\ell(x^{(\ell)})$, where $C_\ell(z),\ z \ge 0$, is a non-decreasing convex continuously differentiable function with $C_\ell(0) = 0$. (For a general discussion of such link cost functions see the comments accompanying expression (46) in [8], or Section 2 in [9].) To be specific, suppose that congestion cost has the meaning of average packet (queueing) delay at the link, for example having form $C_\ell(z) = \frac{a_\ell}{c_\ell - z}$, where c_ℓ is the link (constant) capacity and $a_\ell > 0$ is a fixed parameter.

Finally, the "utility" of flow n is $H_n(x_n)$, where $H_n(z),\ z \ge 0$, is a continuously differentiable concave function. (Again, see [8, 9] for a rationale for the utility function concavity assumption.) The problem is to allocate average flow rates $x = (x_1, \dots, x_N)$ in a way such that the total utility of all flows is maximized subject to some pre-defined upper bounds on the average packet delays along each route. Namely, the control problem is to

$$\text{maximize} \sum_n H_n(x_n) \tag{7}$$

subject to

$$G_n(x) \doteq \sum_{\ell\in\mathcal{R}^{-1}(n)} C_\ell\left(x^{(\ell)}\right) - d_n \le 0, \quad n \in \mathcal{N}. \tag{8}$$

In other words, the problem is to maximize network utility subject to (quality of service) constraints (8) defined in terms of average end-to-end flow delays.

So far, we did not specify the model for how traffic is sent over wireless links from users to their corresponding access points. This model is same as the opportunistic scheduling model described in Section 3.1, except we do not consider power consumption (and the traffic is going from users to access points, not vice versa); formally, the model is as follows. Denote by $\mathcal{N}_i$ the subset of users sending traffic via access point i. (Given our assumptions, sets $\mathcal{N}_i$ for different i do not intersect, and $\cup_i \mathcal{N}_i = \mathcal{N}$.) We assume that the time is slotted, indexed by $t = 0, 1, 2, \dots$, and use slot duration as the time unit. The state $m_i(t)$ of the wireless channel between access point i and the set $\mathcal{N}_i$ of "its" users follows an independent irreducible aperiodic Markov chain with the finite state space M_i. Each access point $i \in \mathcal{I}$ schedules data transmissions of users in $\mathcal{N}_i$ independently of other access points. When the channel state (of access point i) is $m_i \in M_i$, the available scheduling decisions k_i form a finite set $K_i(m_i)$. If access point i chooses decision $k_i \in K_i(m_i)$ in time slot t, then in this slot each node $n \in \mathcal{N}_i$ sends the amount of data (number of bytes) $b_n(k_i) \ge 0$ to the access point.

The convex optimization problem underlying control problem (7)–(8) is as follows:

$$\max_{x\in V} \sum_n H_n(x_n) \tag{9}$$

subject to

$$G_n(x) \le 0, \quad n \in \mathcal{N}, \tag{10}$$

where $V \subset R^N$ is the region of all possible vectors of (long-term) average rates at which flows can be transmitted (over radio channels) to their corresponding access points. The region V is a convex compact set.

Let us compare the control problem (7)–(8) with the extensively studied problem (originally posed in [8]) of maximizing network utility (7) subject to *link capacity constraints*

$$x^{(\ell)} \le c_\ell, \quad \ell \in \mathcal{L}. \tag{11}$$

There are two substantial differences. First, the statement of problem (7) and (11) usually assumes that traffic sources are able to generate traffic at any rate at any time, independently of each other. (Therefore, the convex optimization problem underlying (7) and (11) is: $\max_{x\in R_+^N} \sum_n H_n(x_n)$

subject to (11).) In our model, analogously to the model of Section 3.1, the instantaneous rates at which sources can transmit data are dependent on each other and on the random state of the corresponding channel. Consequently, as already discussed in Section 3.2, the region V in (9) is defined only implicitly, which typically precludes the use of standard convex optimization techniques for a dynamic control. (It is important to emphasize that the region V here defines implicit constraints on joint average rates at which traffic can possibly be *injected* by the sources into the network, *before* the network link capacity constraints (11) or end-to-end delay constraints (8).) If the constraints (8) would be linear, as they are in (11), this difficulty can be resolved by using the GPD algorithm of [15]. However, and this is the second difference from the problem (7) and (11), the constraints (8) are convex non-linear, which requires the extended GPD algorithm of this paper. An asymptotically optimal distributed algorithm for (7)–(8) will be given in Section 7.2.

4 Greedy primal-dual dynamic system

4.1 Optimization problem

Consider a convex compact subset $V \subset R^N$ of a finite-dimensional space R^N, $N \geq 1$. (We will use notation $\mathcal{N} \doteq \{1, \ldots, N\}$ for the set of indices of vectors $\xi = (\xi_1, \ldots, \xi_N) \in R^N$.) Assume that $V \subset \tilde{V}$, where $\tilde{V} \subseteq R^N$ is open and convex, and we have a concave continuously differentiable ("utility") function $H(v)$, $v \in \tilde{V}$.

Consider the following optimization problem:

$$\max_{v \in V} H(v) \tag{12}$$

subject to

$$G_j(v) \leq 0, \quad \forall j \in \mathcal{J}, \tag{13}$$

where $\mathcal{J} \equiv \{1, \ldots, J\}$ is a finite set of indices, and each $G_j(v)$, $v \in \tilde{V}$, is a convex continuously differentiable function.

The problem (12)–(13) can be equivalently written as

$$\max_{v \in V^{cond}} H(v),$$

where

$$V^{\text{cond}} \doteq V \cap \{v \in \tilde{V} \mid G_j(v) \leq 0, \forall j \in \mathcal{J}\}.$$

Clearly, V^{cond} is a convex compact set, when non-empty.

Optimization problem (12)–(13) is feasible when

$$V^{\text{cond}} \neq \emptyset, \tag{14}$$

in which case we denote by $V^* \subseteq V$ the convex compact subset of its optimal solutions. (If H is *strictly* concave, the optimal solution is unique.) Also, we denote by Q^* the convex closed set of optimal solutions $q^* \in R_+^J$ to the following convex optimization problem, dual to the problem (12)–(13):

$$\min_{y \in R_+^J} \left[\max_{v \in V} (H(v) - y \cdot G(v)) \right], \tag{15}$$

where we used notation

$$G(v) \doteq (G_1(v), \ldots, G_J(v)).$$

Next, in Section 4.2, we define a dynamic system, which, as we will show, "solves" problem (12)–(13) under the following non-degeneracy assumption, which is slightly stronger than (14):

$$V \cap \{v \in \tilde{V} \mid G_j(v) < 0, \ \forall j \in \mathcal{J}\} \neq \emptyset. \tag{16}$$

The dynamic system "solves" (12)–(13) in the sense that (assuming (16)) its trajectories converge to the (saddle) set $V^* \times Q^*$.

Note that, under assumption (16), set Q^* is compact. Indeed, the optimal value of the problem (12)–(13) is

$$H(v^*) = \max_{v \in V} (H(v) - q^* \cdot G(v)) \tag{17}$$

for any $v^* \in V^*$ and any $q^* \in Q^*$. Set Q^* must be bounded, because otherwise, given condition (16), we could make the RHS of (17) arbitrarily large by choosing $q^* \in Q^*$ with large norm $\|q^*\|$.

4.2 Dynamic system definition

We define a *trajectory of the greedy primal-dual dynamic system*, or *GPD-trajectory*, as a pair of absolutely continuous functions $(x, q) = ((x(t),\ t \geq 0), (q(t),\ t \geq 0))$, with $x(t)$ taking values in R^N and $q(t)$ taking values in R^J, satisfying the following conditions:

(i) For all $t \geq 0$,

$$x(t) \in \tilde{V}, \tag{18}$$

and for almost all $t \geq 0$,

$$x'(t) = v(t) - x(t), \tag{19}$$

where

$$v(t) \in \arg\max_{v \in V} \left[\nabla H(x(t)) - \sum_j q_j(t) \nabla G_j(x(t)) \right] \cdot v. \tag{20}$$

(ii) We have

$$q(0) \geq 0, \quad (21)$$

$$q_j(t) \geq 0, \forall t \geq 0, \text{ and } q_j'(t) = [G_j(x(t)) + \nabla G_j(x(t)) \cdot (v(t) - x(t))]^+_{q_j(t)} \text{ a.e. in } t \geq 0, \quad j \in \mathcal{J}. \quad (22)$$

The above definition of GPD-trajectories is a generalization of the corresponding definition in [15], which was restricted to the case when all $G_j(\cdot)$ were linear. (In fact, in [15], there are exactly N constraint functions $G_n(v) = v_n$, one for each primal variable v_n. However, extension to an arbitrary finite set of linear constraint functions $G_j(\cdot)$ is just a matter of introducing new variables, as in (26) below.) The interpretation of the dynamic system is analogous to that given in Section 3.2 of [15]: functions $x(t)$ and $q(t)$ are "dynamically changing" primal and dual variables, respectively, for the problem (12)–(13). The term "greedy" refers to condition (20), which states that "control" $v(t)$ is always chosen within the "set of controls" V, so as to greedily maximize $\nabla_x[H(x(t)) - \sum_j q_j(t)G_j(x(t))] \cdot x'(t)$, i.e., the partial time derivative of the Lagrangian $H(x(t)) - \sum_j q_j(t)G_j(x(t))$ *with respect to primal variables $x(t)$ only*. The queueing interpretation given in [15] is basically also valid. The key generalization is the expression for the derivative $q_j'(t)$ of a dual variable (queue length) in (22). Assuming $q_j(t) > 0$ for simplicity, the derivative $q_j'(t)$ is *not* $G_j(v(t))$, but rather the value at point $v(t)$ of the first order (linear) approximation of function $G_j(\cdot)$ about point $x(t)$. *Given the form* (22) *of the derivatives of dual variables (queue lengths) $q_j(t)$*, an alternative interpretation of condition (20) is that control $v(t)$ always greedily maximizes the time derivative of function $H(x(t)) - (1/2)\sum_j q_j^2(t)$, *with respect to both $x(t)$ and $q(t)$*. (See Lemma 1.)

4.3 Global attraction property of GPD-trajectories

The following theorem, showing that GPD-trajectories are such that $(x(t), q(t))$ is attracted to the saddle set $V^* \times Q^*$, is the main result of this paper.

Theorem 1. *Under the non-degeneracy condition* (16)*, the following holds.*

(i) For any GPD-trajectory (x, q), as $t \to \infty$,

$$x(t) \to V^*, \quad (23)$$

$$q(t) \to q^* \text{ for some } q^* \in Q^*. \quad (24)$$

(ii) Let compact subsets $V^\square \subset \tilde{V}$ and $Q^\square \subset R^J_+$ be fixed. Then, the convergence

$$(x(t), q(t)) \to V^* \times Q^* \text{ as } t \to \infty, \quad (25)$$

of GPD-trajectories is uniform with respect to initial conditions $(x(0), q(0)) \in V^\square \times Q^\square$.

Theorem 1 is a generalization of Theorem 2 of [15] in that it allows *non-linear* constraint functions $G_j(\cdot)$. We note that, *in the case when all $G_j(\cdot)$ are linear*, our Theorem 1 is equivalent to Theorem 2 of [15]. To see this, it is sufficient to introduce new variables

$$v_{N+j} \equiv G_j(v_1, \ldots, v_N), \quad j = 1, \ldots, J, \quad (26)$$

replace set $\tilde{V}$ with the extended set $\tilde{V}^{\text{ext}} = \tilde{V} \times R^J$, extend H in the natural way to be a function of $v \in \tilde{V}^{\text{ext}}$, and replace V with set

$$V^{\text{ext}} = \{v = (v_1, \ldots, v_N, v_{N+1}, \ldots, v_{N+J}) \mid (v_1, \ldots, v_N) \in V\} \subset \tilde{V}^{\text{ext}}.$$

Then, the optimization problem (12)–(13) is equivalent to the problem

$$\max_{v \in V^{\text{ext}}} H(v) \quad (27)$$

subject to

$$v_{N+j} \leq 0, \; j = 1, \ldots, J. \quad (28)$$

It is easy to verify that vectors $v^* \in V^{\text{ext}}$ and $q^* \in R^J_+$ are a pair of optimal solutions to the problem (27)–(28) and its dual if and only if $(v_1^*, \ldots, v_N^*) \in V^*$ and $q^* \in Q^*$, where V^* and Q^* are the optimal sets for (12)–(13) and its dual. However, problem (27)–(28) and its corresponding GPD-trajectories are within the framework of Theorem 2 of [15], as shown in [15], Section 3.8.1. Since we have the obvious one-to-one correspondence between GPD-trajectories (as defined in this paper) for the problem (12)–(13) (with all $G_j(\cdot)$ linear) and GPD-trajectories (as defined in [15]) for the problem (27)–(28), the desired equivalence is established.

5 Proof of Theorem 1

The outline of this section is as follows. First, in Section 5.1, we establish some basic properties of the family of GPD-trajectories (including their existence), which hold regardless of conditions (14) and (16); we also make a simple observation (using function F defined in (31)) that, under condition (16), trajectories $(q(t),\ t \geq 0)$ remain bounded. Section 5.2 contains the key proof of statement (i), outlined at the beginning of that section. Given statement (i) of the theorem, the proof of statement (ii) repeats the corresponding proof in [15], with a minor adjustment specified in Section 5.3.

5.1 Basic properties of GPD-trajectories

Unless specified otherwise, throughout this Section 5.1 we do *not* assume condition (14) (or condition (16)).

The following theorem summarizes basic properties of GPD-trajectories, and is a generalization of Lemmas 12 and 13 of [15].

Theorem 2.

(i) *For any $x(0) \in \tilde{V}$ and any $q(0) \in R_+^J$, there exists a GPD-trajectory (x, q) having $(x(0), q(0))$ as initial condition.*

(ii) *The set of GPD-trajectories (x, q) is such that, for arbitrary compact convex set $V^\square$, $V \subseteq V^\square \subset \tilde{V}$, uniformly on $x(0) \in V^\square$, both x and q are Lipschitz continuous, and in addition $x(t) \in V^\square$ for all $t \geq 0$.*

(iii) *The set of GPD-trajectories is closed (in the topology of u.o.c. convergence).*

(iv) *Let compact sets $V^\square \subset \tilde{V}$ and $Q^\square \subset R_+^J$ be fixed. Then, the set of the GPD-trajectories with $x(0) \in V^\square$ and $q(0) = f(0) \in Q^\square$ is compact.*

(v) *If (x, q) is a GPD-trajectory, then, for any $\tau \geq 0$, its shifted (to the left) version $\Theta_\tau(x, q)$ is also a GPD-trajectory. (Formally, $[\Theta_\tau(x, q)](t) = (x(\tau + t), q(\tau + t)),\ t \geq 0$.)*

Proof of Theorem 2 follows closely the development in Section 3.6 of [15], with some adjustments, which we describe here.

Let us denote by $L_{R^N}(0)$ the subset of $C_{R^N}[0, \infty)$, consisting of Lipschitz continuous functions f such that $f(0) = 0$ and $f'(t) \in V$ almost everywhere. We define operator $\mathcal{A}_1$, as operator which takes $(f, x(0), q(0)) \in L_{R^N}(0) \times \tilde{V} \times R_+^J$ into $\mathcal{A}_1(f, x(0), q(0)) = (x, q) \in C_{R^N}[0, \infty) \times C_{R_+^J}[0, \infty)$. The image function x is defined (the same way as in [15]) as the unique solution of the differential equation

$$x'(t) = f'(t) - x(t),\ t \geq 0,\ \text{a.e.},$$

with initial condition $x(0)$. The components of function $q \in C_{R_+^J}[0, \infty)$ are defined (more generally than in [15]) via $q(0)$ and the image function x as follows:

$$q_j(t) = \psi_j(t) - \left[0 \wedge \inf_{0 \leq \xi \leq t} \psi_j(\xi)\right], \quad t \geq 0,\ j \in \mathcal{J}, \tag{29}$$

where

$$\psi_j(t) = q_j(0) + \int_0^t [G_j(x(\xi)) + \nabla G_j(x(\xi)) \cdot (f'(\xi) - x(\xi))] d\xi$$
$$= q_j(0) + \int_0^t G_j(x(\xi)) d\xi + [G_j(x(t)) - G_j(x(0))].$$

Let us denote $C_{\tilde{V}}[0, \infty) \doteq \{x \in C_{R^N}[0, \infty) \mid x(t) \in \tilde{V},\ t \geq 0\}$. We define multivalued operator $\mathcal{A}_2$, which takes $(x, q) \in C_{\tilde{V}}[0, \infty) \times C_{R_+^J}[0, \infty)$ to the set $\mathcal{A}_2(x, q) \subseteq L_{R^N}(0)$, as follows: function $f \in \mathcal{A}_2(x, q)$ if and only if $f \in L_{R^N}(0)$ and

$$f'(t) \in \arg\max_{v \in V} \left[\nabla H(x(t)) - \sum_j q_j(t) \nabla G_j(x(t))\right] \cdot v,$$
$$t \geq 0,\ \text{a.e.}.$$

Clearly, (x, q) is a GPD-trajectory if and only if $\mathcal{A}_1(f, x(0), q(0)) = (x, q)$ and $f \in \mathcal{A}_2(x, q)$ for some f. This in turn is equivalent to the existence of a fixed point f of the operator $\mathcal{A}_2 \mathcal{A}_1(f, x(0), q(0))$. Such fixed points in fact exist, which is shown using Kakutani theorem, analogously to the way it is done in [15]. This proves statement (i).

Statements (ii)–(v) are proved completely analogously to the proof of the corresponding statements of Lemmas 12–13 of [15]. In particular, the proof of (ii) relies in an essential way on the fact that (by Proposition 1 in the Appendix), for any GPD-trajectory (x, q),

$$\rho(x(t), V) \leq \rho(x(0), V) e^{-t}, \quad t \geq 0, \tag{30}$$

and $x(t)$ for all $t \geq 0$ is contained within the (compact) convex hull of $V \cup \{x(0)\}$. □

Let us call a time point $t > 0$ *regular* (for given GPD-trajectory (x, q)) if proper derivatives $x'(t)$ and $q'(t)$ exist, and conditions (19), (20), (22) hold for this t. Almost all $t \geq 0$ are regular. To simplify notation, throughout the rest of this entire Section 5 we adopt a convention that any expression or statement involving any of the functions $x'(t)$, $q'(t)$, or $v(t)$, holds under the additional assumption that t is a regular point, even if we do not state it explicitly.

Let us introduce the following function:

$$F(v, y) = H(v) - \frac{1}{2} \sum_{j \in \mathcal{J}} y_j^2, \quad v \in \tilde{V},\ y \in R_+^J. \tag{31}$$

Lemma 1. *For any GPD-trajectory, at any (regular) $t \geq 0$,*

$$\frac{d}{dt}F(x(t), q(t)) = \nabla H(x(t)) \cdot (v(t) - x(t)) - \sum_j q_j(t)[G_j(x(t)) + \nabla G_j(x(t)) \cdot (v(t) - x(t))] \quad (32)$$

and

$$v(t) \in \arg\max_{v \in V} \nabla H(x(t)) \cdot (v - x(t)) - \sum_j q_j(t)[G_j(x(t)) + \nabla G_j(x(t)) \cdot (v - x(t))]. \quad (33)$$

(*Expressions* (32) *and* (33) *imply that, given $x(t)$ and $q(t)$, $v(t)$ is a point in V maximizing $(d/dt)F(x(t), q(t))$.*)

If, in addition, (14) *holds* (*that is, V^* is non-empty*), *then for any $v^* \in V^*$*

$$\frac{d}{dt}F(x(t), q(t)) \geq \nabla H(x(t)) \cdot (v^* - x(t)) \geq H(v^*) - H(x(t)). \quad (34)$$

Proof: Expression (32) follows directly from (19) and (22). Inclusion (33) is equivalent to (20). The first inequality in (34) follows from condition (33) and the fact that (by subgradient inequality)

$$G_j(x(t)) + \nabla G_j(x(t)) \cdot (v^* - x(t)) \leq G_j(v^*) \leq 0,$$

and the second one is again the subgradient inequality. □

Lemma 2. *Under non-degeneracy condition* (16), *for any compact $V^\square \subset \tilde{V}$ and any compact $Q^\square \subset R_+^J$, uniformly on the GPD-trajectories with $(x(0), q(0)) \in V^\square \times Q^\square$,*

$$\sup_{t \geq 0} \|q(t)\| < \infty.$$

Proof is essentially same as that of Lemma 4 in [15]. The key point here is that, if we pick arbitrary $v' \in V$ such that $G_j(v') < 0$ for all j, we have (by (32)–(33))

$$\frac{d}{dt}F(x(t), q(t)) \geq \nabla H(x(t)) \cdot (v' - x(t)) - \sum_j q_j(t)[G_j(x(t)) + \nabla G_j(x(t)) \cdot (v' - x(t))] \geq \nabla H(x(t)) \cdot (v' - x(t)) - \sum_j q_j(t) G_j(v'),$$

and therefore, since $x(t)$ stays within a compact set (by Theorem 2(ii)), function $F(x(t), q(t))$ is strictly increasing when $\|q(t)\|$ is large. □

5.2 Proof of Theorem 1(i)

Throughout this Section 5.2, we always assume that non-degeneracy condition (16) holds.

The outline of the proof is analogous to that given in Section 3.5 of [15]. First, in Lemma 3 we give a characterization of optimal dual solutions $q^* \in Q^*$, convenient for our purposes. According to Lemma 1, "control" $v(t)$ is always chosen within the "set of controls" V so as to maximize the derivative of $F(x(t), q(t))$. However, the key difficulty is that $F(x(t), q(t))$ is *not* necessarily non-decreasing along a GPD-trajectory. To overcome this, we introduce a different function F^*, defined in (37), which happens to be non-decreasing as long as $x(t) \in V$. (Although $x(t) \in V$ does not necessarily hold, we know from (30) that $x(t)$ converges to V, and this convergence is fast. This, roughly speaking, "reduces" our situation to the case $x(t) \in V$.) Using F^* allows us to prove in the key Lemma 6 the convergence of $x(t)$ to a set $V^{\max} \subseteq V$, which contains V^*, and the convergence of $q(t)$ to some point within Q^*. (The proof of Lemma 6 is more involved than the corresponding proof in [15]. In particular, we need to consider additional component $B_4(t)$ of the derivative of $F^*(x(t), q(t))$ and establish (43) to prove Lemma 6(v).) In Lemma 7, using the fact that $q(t)$ converges, we prove that $x(t) \to R_-^+$, which implies $x(t) \to V^{\text{cond}} = V \cap R_-^N$. Finally, in Lemma 8 we prove the convergence $x(t) \to V^*$, using the fact that function $F(x(t), q(t))$ is non-decreasing when $x(t) \to V^{\text{cond}}$.

Recall that, under condition (16), both optimal sets V^* and Q^* are compact. By Kuhn-Tucker theorem, for any pair of optimal primal and dual solutions, $v^* \in V^*$ and $q^* \in Q^*$, the complementary slackness condition holds:

$$q^* \cdot G(v^*) = 0. \quad (35)$$

Note that there always exists $v^* \in V^*$ for which the subset of $j \in \mathcal{J}$ with $G_j(v^*) < 0$ is the maximum possible; this subset we denote by $\mathcal{J}^{(0)} \subseteq \mathcal{J}$. Thus, a vector $q^* \in R_+^J$ satisfies the complementary slackness condition (35) *for all* $v^* \in V^*$ if and only if

$$q_j^* = 0 \text{ for } j \in \mathcal{J}^{(0)}. \quad (36)$$

For an optimal point $v^* \in V^*$, let $C^*(v^*)$ denote the normal cone to V at v^*. (It may have any dimension from 0 to N. A zero-dimensional cone is the one containing the single vector 0 - this is the case when v^* lies in the interior of V.) We know that $(v^*, q^*) \in V^* \times Q^*$ if and only if (v^*, q^*) is a saddle point of the Lagrangian $H(v) -$

$y \cdot G(v)$, $v \in V$, $y \in R^J_+$, for the pair of primal problem (12)–(13) and its dual (15). This implies the following property, recorded here for future reference.

Lemma 3. *Assume non-degeneracy condition* (16). *Then, the following holds for any fixed* $v^* \in V^*$. *Vector* $q^* \in Q^*$ *if and only if* $q^* \in R^J_+$, $\nabla H(v^*) - \sum_{j \in J} q^*_j \nabla G_j(v^*) \in C^*(v^*)$ *and the complementary slackness condition* (35) (*or, condition* (36)) *holds.*

Let us fix arbitrary optimal dual solution $q^* \in Q^*$, and associate with it the following function

$$F^*(v, y) = H(v) - q^* \cdot G(v) - \frac{1}{2}\sum_{j \in \mathcal{J}}(y_j - q^*_j)^2,$$
$$v \in \tilde{V}, \quad y \in R^J_+. \tag{37}$$

As will be shown below in Lemmas 5 and 6, the function $F^*(x(t), q(t))$ is "asymptotically non-decreasing" along GPD-trajectories. (If $x(0) \in V$, it is in fact non-decreasing. See the remark following the proof of Lemma 6.) This property makes it the key "tool" in proving Theorem 1(i).

We also denote

$$H^*(v) \doteq H(v) - q^* \cdot G(v)\ ,$$

and so $F^*(v, y) = H^*(v) - (1/2)\sum_n (y_n - q^*_n)^2$. Function $H^*(v)$ is the Lagrangian $H(v) - y \cdot G(v)$ of the problem (12)–(13), with the dual variable y equal to $q^* \in Q^*$. This implies that the convex compact set

$$V^{\max} \doteq \arg\max_{v \in V} H^*(v)$$

contains all optimal solutions to the problem (12)–(13), i.e. $V^* \subseteq V^{\max}$, and $H^*(v^*) = H(v^*)$ for any $v^* \in V^*$. Within $V^{\max}$, function $H^*(v)$ is constant and therefore $H(v)$ is linear. (This is because if either $H(\cdot)$ or at least one of the functions $G_j(\cdot)$ with strictly positive q^*_j would not be linear within the convex set $V^{\max}$, function $H^*(v)$ could not be constant—in fact could not be linear—on $V^{\max}$; recall that $-H(\cdot)$ and all $G_j(\cdot)$ are convex.) Then, by Proposition 2 (in Appendix), $\nabla H(v)$ is constant within $V^{\max}$. For future reference, we record these facts in the following lemma.

Lemma 4. *For arbitrary fixed optimal dual solution* $q^* \in Q^*$ *and its associated function* $H^*(\cdot)$ *and set* $V^{\max}$, *we have:*

(i) $V^{\max} \supseteq V^*$,
(ii) $\nabla H(v)$ *is constant within* $V^{\max}$,
(iii) $H^*(v^*) = H(v^*)$ *for any* $v^* \in V^*$.

Now we can proceed with the main part of the proof of Theorem 1(i). From this point on in this Section 5.2 we consider a fixed GPD-trajectory (x, q).

Lemma 5. *Consider* $F^*(\cdot, \cdot)$ *associated with arbitrary* $q^* \in Q^*$. *Then, for all (regular)* $t \geq 0$,

$$\frac{d}{dt}F^*(x(t), q(t)) \geq Y(x(t), q(t), v(t)) \tag{38}$$

where, for $v \in V$,

$$\begin{aligned} &Y(x(t), q(t), v)\\ &\quad \doteq \nabla H(x(t)) \cdot (v(t) - x(t))\\ &\qquad -\left[\sum_j q^*_j \nabla G_j(x(t))\right] \cdot (v - x(t))\\ &\qquad -\sum_j \big(q_j(t) - q^*_j\big)[G_j(x(t)) + \nabla G_j(x(t)) \cdot (v - x(t))]. \end{aligned}$$

Consequently, for any (regular) $t \geq 0$,

$$v(t) \in \arg\max_{v \in V} Y(x(t), q(t), v). \tag{39}$$

Proof: Inequality (38) is obtained by writing out the expression for $\frac{d}{dt}F^*(x(t), q(t))$, and observing that $q_j(t) = 0$ implies $-(q_j(t) - q^*_j) \geq 0$. Inclusion (39) follows from the definition of a GPD-trajectory, because the term in $Y(x(t), q(t), v)$ involving v is

$$\left[\nabla H(x(t)) - \sum_j q_j(t) \nabla G_j(x(t))\right] \cdot v. \qquad \square$$

Lemma 6. *Consider* $F^*(\cdot, \cdot)$ *associated with arbitrary fixed* $q^* \in Q^*$. *Let* $v^* \in V^*$ *be fixed and such that* $G_j(v^*) < 0$ *for each* $j \in \mathcal{J}^{(0)}$. *The following properties hold, as* $t \to \infty$.

(i) $x(t) \to V^{\max}$. *Consequently, by Lemma 4(i)–(ii),* $\nabla H(x(t)) \to \nabla H(v^*)$.
(ii) $q_j(t) \to 0$ *for every* $j \in \mathcal{J}^{(0)}$.
(iii) $[\nabla H(x(t)) - \sum_j q_j(t) \nabla G_j(x(t))] \to C^*(v^*)$.
(iv) *Both* $F^*(x(t), q(t))$ *and* $H^*(x(t))$ *converge (to some constants). Consequently,* $\sum_n (q_n(t) - q^*_n)^2 = \|q(t) - q^*\|^2$ *converges.*
(v) $q(t) \to q^{**}$, *for some fixed element* $q^{**} \in Q^*$.

Proof: All statements of the lemma will follow from the representation of the lower bound $Y(x(t), q(t), v(t))$ of the derivative $\frac{d}{dt}F^*(x(t),q(t))$. Let us use the following notation:

$$W_j = W_j(x(t), v^*) \doteq \nabla G_j(x(t)) \cdot (v^* - x(t)) - [G_j(v^*) - G_j(x(t))].$$

By the subgradient inequality, for any $j \in \mathcal{J}$ and any $t \geq 0$, $W_j \leq 0$ and, moreover,

$$\nabla G_j(x(t)) \neq \nabla G_j(v^*) \text{ implies } W_j < 0. \tag{40}$$

(This is because $W_j = 0$ implies that $G_j(\cdot)$ is linear along the segment L connecting points $x(t)$ and v^*, which in turn implies, by Proposition 2, that $\nabla G_j(\xi)$ is constant along L.)

Then, we can write:

$$\begin{aligned}
&Y(x(t), q(t), v(t))\\
&\quad = Y(x(t),q(t),v^*) + [Y(x(t),q(t),v(t)) - Y(x(t),q(t),v^*)]\\
&\quad = \nabla H(x(t)) \cdot (v^* - x(t)) - \sum_j q_j(t) W_j\\
&\qquad - \sum_j q_j^*[G_j(v^*) - G_j(x(t))] - \sum_j (q_j(t) - q_j^*) G_j(v^*)\\
&\qquad + [Y(x(t), q(t), v(t)) - Y(x(t), q(t), v^*)]\\
&\quad = B_1(t) + B_2(t) + B_3(t) + B_4(t),
\end{aligned}$$

where

$$\begin{aligned}
B_1(t) &= \nabla H(x(t)) \cdot (v^* - x(t)) - \sum_j q_j^*[G_j(v^*) - G_j(x(t))]\\
&\geq H^*(v^*) - H^*(x(t)),\\
B_2(t) &= -\sum_j \left(q_j(t) - q_j^*\right) G_j(v^*) = -\sum_{j \in \mathcal{J}^{(0)}} q_j(t) G_j(v^*) \geq 0,\\
B_3(t) &= Y(x(t), q(t), v(t)) - Y(x(t), q(t), v^*)\\
&= \left[\nabla H(x(t)) - \sum_j q_j(t) \nabla G_j(x(t))\right]\\
&\quad \cdot (v(t) - v^*) \geq 0,\\
B_4(t) &= -\sum_j q_j(t) W_j \geq 0.
\end{aligned}$$

As in [15], we use further breakdown of $B_1(t)$, as follows. We denote by $x^*(t)$ the normal projection of $x(t)$ onto V; namely, $x^*(t)$ is the (unique) point of V which is the closest to $x(t)$. According to (30),

$$\|x(t) - x^*(t)\| \leq \|x(0) - x^*(0)\| e^{-t}, \quad t \geq 0.$$

We have

$$B_1(t) \geq H^*(v^*) - H^*(x(t)) = B_{11}(t) + B_{12}(t),$$

where $B_{11}(t) = H^*(v^*) - H^*(x^*(t)) \geq 0$ and $B_{12}(t) = H^*(x^*(t)) - H^*(x(t))$. For the function $B_{12}(t), t \geq 0$, the following estimate holds for any pair $0 \leq t_1 \leq t_2 \leq \infty$:

$$\begin{aligned}
\int_{t_1}^{t_2} |B_{12}(s)| ds &\leq \int_{t_1}^{t_2} C_1 \|x^*(s) - x(s)\| ds\\
&\leq C_1 \int_{t_1}^{t_2} \|x^*(0) - x(0)\| e^{-s} ds\\
&= C_1 \|x^*(0) - x(0)\| [e^{-t_1} - e^{-t_2}]\\
&\leq C_1 \|x^*(0) - x(0)\| < \infty,
\end{aligned} \tag{41}$$

where $C_1 > 0$ is a uniform upper bound on $\|\nabla H^*(x(s))\|$ and $\|\nabla H^*(x^*(s))\|$ over all $s \geq 0$. (For example, C_1 can be chosen as the maximum of $\|\nabla H^*(\xi)\|$ over all ξ in the convex hull of $V \cup \{x(0)\}$.)

Given the above representation of $Y(x(t), q(t), v(t))$, the proof of statements (i)–(iii) essentially repeats the proof of statements (i)–(iii) of Lemma 6 in [15], using functions B_{11}, B_{12}, B_2 and B_3 (and the fact that B_4 is non-negative). For example, the estimate of $B_3(t)$ (in addition to its non-negativity), takes the following form: for any $\epsilon_1 > 0$ there exists sufficiently small $\epsilon_2 > 0$ such that

$$B_3(t) \geq \epsilon_2 \text{ as long as}$$
$$\rho\left(\left[\nabla H(x(t)) - \sum_j q_j(t) \nabla G_j(x(t))\right], C^*(v^*)\right) \geq \epsilon_1. \tag{42}$$

(And the proof of (42) is same as that of its special case in [15], with $q(t)$ replaced by $\sum_j q_j(t) \nabla G_j(x(t))$.)

Proof of (iv): Convergence of $H^*(x(t))$ follows directly from (i). Function $F^*(x(t), q(t))$ is absolutely continuous (in fact—Lipschitz), bounded, and its derivative $(d/dt)F^*(x(t), q(t)) \geq B_{12}(t)$. The derivative lower bound implies that $F^*(x(t), q(t))$ can be represented as a sum of some non-decreasing function and the non-increasing function

$$\int_0^t [B_{12}(s) \wedge 0] ds.$$

The latter function converges (by (41)), and then $F^*(x(t), q(t))$ converges as well.

To prove (v), consider any limiting point (x^{**}, q^{**}) of the trajectory $(x(t), q(t)), t \geq 0$, which exists as $t \to \infty$, since the trajectory is bounded. We must have $[\nabla H(v^*) -$

$\sum_j q_j^{**}\nabla G_j(x^{**})] \in C^*(v^*)$, by (i) and (iii). Using function $B_4(t)$ and property (40), it is easy to observe that (x^{**}, q^{**}) must be such that, for any $j \in \mathcal{J}$,

$$q_j^{**}\|\nabla G_j(x^{**}) - \nabla G_j(v^*)\| = 0. \tag{43}$$

(Otherwise, $F^*(x(t), q(t))$ would go to $+\infty$.) This implies that

$$\left[\nabla H(v^*) - \sum_j q_j^{**}\nabla G_j(v^*)\right]$$
$$= \left[\nabla H(v^*) - \sum_j q_j^{**}\nabla G_j(x^{**})\right] \in C^*(v^*).$$

We also know that $q_j^{**} = 0$ for every $j \in \mathcal{J}^{(0)}$, by (ii). Therefore, by Lemma 3, $q^{**} \in Q^*$. Note that properties (i)–(iv) of this Lemma hold for any a priori fixed $q^* \in Q^*$, including q^{**}. Therefore, by (iv), $\|q(t) - q^{**}\|$ converges, and it can only converge to 0. □

Remark. As seen from the proof of Lemma 6, $x(0) \in V$ implies that $F^*(x(t), q(t))$ is non-decreasing. Indeed, in this case $x(t) \in V$ for all $t \geq 0$ and therefore $B_{12}(t) = H^*(x^*(t)) - H^*(x(t)) \equiv 0$. If $x(0) \notin V$, function $F^*(x(t), q(t))$ is still "asymptotically non-decreasing," due to estimate (41), as shown in the proof of Lemma 6(iv).

Lemma 7. *The following property holds:*

$$\limsup_{t\to\infty} G_j(x(t)) \leq 0, \quad j \in \mathcal{J}.$$

Consequently, as $t \to \infty$, $x(t) \to V^{\max} \cap \{G_j(v) \leq 0, \forall j \in \mathcal{J}\}$.

Proof: From (29), for any $j \in \mathcal{J}$ and any pair $0 \leq t_1 \leq t_2 < \infty$, we have the inequality

$$q_j(t_2) - q_j(t_1) \geq \int_{t_1}^{t_2} G_j(x(t))dt + [G_j(x(t_2)) - G_j(x(t_1))].$$

The rest of the proof is analogous to that of Lemma 7 in [15]. □

Lemma 8. *We have* $x(t) \to V^*$ *as* $t \to \infty$.

Proof: is analogous to that of Lemma 8 in [15]. □

The proof of Theorem 1(i) is complete.

5.3 Proof of Theorem 1(ii)

This proof repeats that of Theorem 2 in [15] virtually verbatim. (In two places in the proof of Lemma 16 in [15], R_-^N has to be replaced by $\{v \in R^N \mid G_j(v) \leq 0, \ \forall j \in \mathcal{J}\}$.)

6 A dynamic resource allocation model

In this section we define and study a resource allocation model such that each control action "generates" certain amounts of commodities. The "utility" of the system (under a given control strategy) is a concave function H of the average rates at which commodities are generated. The problem is to find a control strategy which maximizes utility subject to a number of given convex constraints on the vector of average commodity generation rates. In Sections 6.1–6.3 we introduce the model and the optimization problem formally. We define a dynamic control policy, called Greedy-Primal Dual (GPD) algorithm, in Section 6.4. (This algorithm is a generalization of the GPD algorithm, introduced in [15], which was applicable to the problem with *linear* constraints.) In Section 6.5 we prove asymptotic optimality of the algorithm (as one of its parameters approaches 0). Finally, in Section 6.6, we present the extension of the GPD algorithm, such that it (asymptotically) solves the above problem for the more general network model of [15].

6.1 The model

We consider a system consisting of a finite set of *nodes* $\mathcal{N}^u = \{1, 2, \ldots, N_u\}$, $N_u \geq 1$. (In the terminology of [15], these are "utility" nodes.) The system operates in discrete time $t = 0, 1, 2, \ldots$ as follows. (By convention, we will identify an integer time t with the unit time interval $[t, t+1)$, which will sometimes be referred to as the *time slot* t.) The system has a finite set of *modes* M. The sequence of modes $m(t)$, $t = 0, 1, 2, \ldots$, forms an irreducible (finite) Markov chain with stationary distribution $\{\pi_m, \ m \in M\}$, where all $\pi_m > 0$ and $\sum \pi_m = 1$. (The mode process $m(t)$ models the underlying randomly changing system "environment," and is *not* affected by any network control.) When the network mode is $m \in M$, a finite number of *controls* is available, which form set $K(m)$. (We denote by $K \doteq \cup_m K(m)$ the finite set of all possible controls across all modes $m \in M$.) When a control $k \in K(m)$ is chosen at time t, each node $n \in \mathcal{N}^u$ generates an amount $b_n(k)$ of certain commodity. We will denote $b(k) \doteq (b_1(k), \ldots, b_{N_u}(k))$.

Informally, the problem we are going to address is as follows. Let $x^* = (x_1^*, \ldots, x_{N_u}^*)$ denote the average value of $b(k(t))$ under a given dynamic control policy. We would like to find a dynamic control policy which maximizes some concave utility function $H(x^*)$, subject to the finite number of constraints

$$G_j(x^*) \le 0, \quad j \in \mathcal{J}, \tag{44}$$

where all $G_j(\cdot)$ are convex.

Remark. The *problem* we just (informally) described is more general than that considered for the network control model in [15], in that here we have additional—possibly non-linear—constraints (44). However, the system *model* defined above is a special case of that in [15]. In Section 6.6 we will show that all results of this Section 6 in fact easily generalize for the (more general) network model of [15].

6.2 System rate region

In this section we define the system rate region $V \subset R^{N_u}$, which is the set of all possible long-term average values of vector $b(k(t))$, where $k(t)$ is control chosen at time t. Formally, the definition is as follows.

Suppose, for each network mode $m \in M$, a probability distribution $\phi_m = (\phi_{mk}, k \in K(m))$ is fixed, which means that $\phi_{mk} \ge 0$ for all $k \in K(m)$, and $\sum_{k \in K(m)} \phi_{mk} = 1$. For such a set of distributions $\phi \doteq (\phi_m,\ m \in M)$, consider the following vector

$$v(\phi) = \sum_{m \in M} \pi_m \sum_{k \in K(m)} \phi_{mk} b(k)\ .$$

If we interpret ϕ_{mk} as the long-term average fraction of time slots when control $k \in K(m)$ is chosen among the slots when the network mode is m, then $v(\phi)$ is the corresponding vector of long-term average rates at which commodities are generated. The system *rate region* V is defined as the set of all possible vectors $v(\phi)$ corresponding to all possible ϕ. Clearly, V is a convex compact (in fact—polyhedral) subset of R^{N_u}, as a linear image of the compact polyhedral set of all possible values of ϕ. Rate region V may turn out to be degenerate (i.e., have dimension less than N_u).

6.3 The underlying optimization problem

Let us denote by $B \subseteq R^{N_u}$ the convex hull of the set $\{b(k),\ k \in K\}$. (Clearly, B is convex compact and $B \supseteq V$.) Suppose an open convex set $\tilde{V}$, $V \subseteq B \subset \tilde{V} \subseteq R^{N_u}$, is fixed.

Suppose a continuously differentiable concave *utility function* $H(v)$ is defined on $\tilde{V}$. Consider the following optimization problem:

$$\max_{v \in V} H(v) \tag{45}$$

subject to

$$G_j(v) \le 0, \quad j \in \mathcal{J}, \tag{46}$$

where $\mathcal{J} = \{1, 2, \ldots, J\}$ is a finite set and each $G_j(\cdot)$ is a continuously differentiable convex function on $\tilde{V}$. Problem (45)–(46) is feasible when

$$V \cap \{v \in \tilde{V} \mid G_j(v) \le 0,\ \forall j \in \mathcal{J}\} \ne \emptyset, \tag{47}$$

in which case we denote by V^* the compact convex set of optimal solutions of (45)–(46), and by $Q^* \subseteq R_+^J$ the closed convex set of optimal solutions to the problem dual to (45)–(46).

We seek to find a dynamic control algorithm, such that, when problem (45)–(46) is feasible, the corresponding average commodity rates $x^* \in V^*$.

In the next Section 6.4 we introduce an algorithm (called GPD algorithm), which is (asymptotically) optimal in the sense that it (asymptotically) achieves the goal described above, under the following non-degeneracy assumption, which is slightly stronger than feasibility condition (47):

$$V \cap \{v \in \tilde{V} \mid G_j(v) < 0,\ \forall j \in \mathcal{J}\} \ne \emptyset. \tag{48}$$

(Under (48), Q^* is a compact set, as well as V^*. See Section 4.1.)

Remark. Non-degeneracy assumption (48), or even a weaker feasibility assumption (47), are *not* needed for *any* of the results of this Section 6, which are concerned with system dynamics under GPD algorithm. Assumption (48) is only invoked to apply Theorem 1 (which says that the dynamic system in fact converges to an optimal state), and thus establish asymptotic optimality of the GPD algorithm. (See the beginning of Section 6.5.)

6.4 Greedy primal-dual algorithm

Consider the following control policy. (Recall that $H(\cdot)$ is the utility function defined in Section 6.3.)

Greedy primal-dual (GPD) algorithm. *At time t choose a control*

$$k(t) \in \arg\max_{k \in K(m(t))} \left[\nabla H(X(t)) - \sum_{j \in \mathcal{J}} \beta Q_j(t) \nabla G_j(X(t))\right] \cdot b(k), \tag{49}$$

where running average $X(t)$ of the values of vector $b(k(t))$ for utility nodes is updated as follows:

$$X(t+1) = (1-\beta)X(t) + \beta b(k(t)), \tag{50}$$

with $\beta > 0$ being a (small) parameter, and where the "virtual queue lengths" $Q_j(t)$, $j \in \mathcal{J}$, are updated as follows:

$$Q_j(t+1) = [Q_j(t) + G_j(X(t)) + \nabla G_j(X(t)) \cdot (b(k(t)) - X(t))]^+. \quad (51)$$

The initial values $X(0) \in \tilde{V}$ and $Q_j(0) \geq 0$, $j \in \mathcal{J}$, are fixed arbitrarily.

Remark. The initial condition $X(0) \in \tilde{V}$ and the update rule (50) imply (by induction) that $X(t) \in \tilde{V}$ for all $t \geq 0$. Therefore, the (random) system evolution is well defined for all $t \geq 0$, because all the functions in (49) and (51) are well defined.

We will use notation $Q(t) = (Q_1(t), \ldots, Q_J(t))$. It follows from (51) that $Q(t) \in R^J_+$ for all t.

6.5 Asymptotic optimality of GPD algorithm

The main result of this section (Theorem 3) is a generalization of Theorem 3 in [15]. It shows that, as $\beta \downarrow 0$, the "fluid-scaled" processes $\{(X(t/\beta),\ t \geq 0),\ (\beta Q(t/\beta),\ t \geq\ 0)\}$ converge to a (generally speaking) random process $\{(x(t), t \geq\ 0),\ (q(t),\ t \geq 0)\}$ with sample paths being GPD-trajectories (as defined in Section 4) for the optimization problem (45)–(46). (This result, as well as all other results of Section 6, does *not* use assumption (48), or even (47).) But, according to Theorem 1, under the non-degeneracy assumption (48), for all GPD-trajectories, as time $t \to \infty$, we have $(x(t), q(t)) \to V^* \times Q^*$, where V^* and Q^* are the sets of optimal solutions of the problem (45)–(46) and its dual, respectively. In this sense, Theorems 3 and 1 demonstrate *asymptotic optimality* of the GPD algorithm. (The *proof* of Theorem 3 is a straightforward extension of that of Theorem 3 in [15], and will be omitted.)

Let us discuss what the asymptotic optimality of the GPD algorithm means in terms of applications. According to the update rule (50), the value of $X(t)$ is the exponentially weighted average of the past values of $b(k(\cdot))$, i.e.,

$$X(t) = \sum_{i=0}^{\infty} \beta(1-\beta)^i b(k(t-i)). \quad (52)$$

Due to simplicity of rule (50), such averaging is widely used in many applications (for example, in wireless systems utilizing opportunistic scheduling—cf. [4, 14, 16]); $X(t)$ has roughly the meaning of the average of the values of $b(k(\cdot))$ within time interval $[t - 1/\beta, t]$. Then, the combination of Theorems 3 and 1 means that commodity rates, *measured as average commodity values over $1/\beta$-long time intervals*, converge (close to) their optimal values *within a time interval of the order of* $1/\beta$. (This point is discussed in more detail and made precise in [14], where the "Gradient" algorithm—a special case of GPD—is analized. We note that, in our case, the avove statement is true as long as initial values of Q_j are bounded). In other words, under the GPD algorithm, from any initial state (as long as Q_j are bounded), within time of the order of $1/\beta$ the system "makes a transition" to an (almost) optimal *regime* in which average (over $1/\beta$-long intervals) commodity rates are (close to) optimal.

Depending on the time scales of the system, the average rate X may in fact be a short-term average, or "instantaneous", rate. Indeed, if the time slot is short, say 1.67 msec (see [4]), and $\beta = 1/600$, then X represents roughly the average rate over a 1 sec interval. In the applications such as opportunistic scheduling in wireless systems, it is typically infeasible for the "true instantaneous rate" (over each scheduling slot) to be close to the optimal average rate, as there is only a discrete set of available instantaneous rates and, moreover, this set depends on the random state of the radio channel. Therefore, in such applications the short-term average rate X, as defined by (52), serves as a reasonable notion of "instantaneous" rate.

6.5.1 Asymptotic regime: Fluid scaled processes

First, we need to define the asymptotic regime formally. From this point on in the paper, we consider a sequence of processes $S^\beta = (X^\beta, Q^\beta, m^\beta)$, indexed by the value of parameter β, with $\beta \downarrow 0$ along a sequence $\mathcal{B} = \{\beta_j,\ j = 1, 2, \ldots\}$ such that $\beta_j > 0$ for all j. The initial state $S^\beta(0) = (X^\beta(0), Q^\beta(0), m^\beta(0))$ is fixed for each $\beta \in \mathcal{B}$, and it satisfies the conditions specified in the GPD algorithm definition in Section 6.4. (Here and below, the processes and variables pertaining to a fixed parameter β will be supplied the upper index β. Expression $\beta \downarrow 0$ means that β converges to 0 along the sequence $\mathcal{B}$, unless otherwise specified.)

The probability law of the Markov chain $m^\beta(\cdot)$ describing the system mode process is same for each β.

Before we introduce fluid-scaled version of the process (for each $\beta \in \mathcal{B}$), we need to augment the definition of the process itself. First, let us extend the definition of $X^\beta(t)$ to continuous time $t \in R_+$ by adopting the convention that $X^\beta(t)$ is constant within each time slot $[l, l+1)$. We do analogous domain extension for $Q^\beta(t)$. Thus, each β, we consider the (continuous time) process (X^β, Q^β), where

$$X^\beta = (X^\beta(t),\ t \geq 0), \quad Q^\beta = (Q^\beta(t),\ t \geq 0).$$

For each β consider the following process (x^β, q^β), which is a *fluid-scaled* version of (X^β, Q^β):

$$x^\beta(t) \doteq X^\beta(t/\beta), \tag{53}$$

$$q^\beta(t) \doteq \beta Q^\beta(t/\beta). \tag{54}$$

Note that all component functions of (x^β, q^β) are piece-wise constant, with the "time slot" of length β.

6.5.2 Fluid scaled processes converge to processes concentrated on GPD-trajectories

We will view *random processes* (x^β, q^β) as processes with realizations in the Skorohod space $D_{R^{N_u+J}}[0,\infty)$ of functions with domain $[0,\infty)$, taking values in R^{N_u+J}, which are right-continuous and have left-limits. The Skorohod topology and corresponding Borel σ-algebra on $D_{R^{N_u+J}}[0,\infty)$ are defined in the usual way. (Cf. [6] for the definitions.)

Theorem 3. *Consider the sequence of processes* $\{(x^\beta, q^\beta)\}$ *with* $\beta \downarrow 0$ *along set* $\mathcal{B}$. *Assume that* $(x^\beta(0), q^\beta(0)) \to (x(0), q(0))$, *where* $(x(0), q(0)) \in \tilde{V} \times R_+^J$ *is a fixed vector. Then, the sequence* $\{(x^\beta, q^\beta)\}$ *is relatively compact and any weak limit of this sequence (i.e., a process obtained as a weak limit of a subsequence of* $\{(x^\beta, q^\beta)\}$) *is a process with sample paths* (x, q) *being with probability* 1 *GPD-trajectories (with initial state* $(x(0), q(0))$) *for the optimization problem* (45)–(46) *(with* $\mathcal{N} = \mathcal{N}^u$, $N = N_u$).

As mentioned above, the proof of Theorem 3 is a straightforward generalization of that of Theorem 3 in [15].

6.6 A more general network model

As we mentioned earlier, all results of this Section 6 can be easily extended for the case where the system *model* of Section 6.1 is replaced by the more general queueing network model of [15]. Next, we specify the more general network model and the corresponding GPD algorithm. After that, we explain how the asymptotic optimality of the generalized algorithm is proved.

The network model of [15] is the model of Section 6.1 augmented as follows. In addition to the N_u "utility" nodes forming set $\mathcal{N}^u = \{1, 2, \dots, N_u\}$, there are also N_p "processing" nodes forming set $\mathcal{N}^p = \{N_u + 1, \dots, N\}$. Each processing node $n \in \mathcal{N}^p$ has associated *queue*, formed by *customers* waiting for *processing* (or *service*) by the node. The corresponding *queue length* at time t is denoted by $Q_n(t)$, $n \in \mathcal{N}^p$. (Recall that the variables $Q_j(t)$ for $j \in \mathcal{J}$ we called "virtual" queue lengths.) If control $k \in K(m)$ is chosen at time t, associated with it (in addition to the generated commodity amounts vector $b(k) = (b_1(k), \dots, b_{N_u}(k)))$ is the following sequence of actions (which occur in the order listed):

(a) each processing node $n \in \mathcal{N}^p$ serves integer number $\mu_n(k) \ge 0$ of customers from its queue (or the entire queue n content, if it is less than $\mu_n(k)$), which are then randomly and independently routed to other processing nodes (including possibly self) with probabilities $p_{n\ell}(k)$, $\ell \in \mathcal{N}^p$, $\sum_{\ell\in\mathcal{N}^p} p_{n\ell}(k) \le 1$, or leave the system with probability $1 - \sum_{\ell\in\mathcal{N}^p} p_{n\ell}(k)$;

(b) an integer number $\lambda_n(k) \ge 0$ of exogenous customers arrive into each processing node queue $n \in \mathcal{N}^p$.
We make a non-restrictive in most applications assumption that if $k \in K(m)$ is a control allowed in mode m, then a "version" of this control, with $\mu_n(k)$ replaced by 0 for any subset of processing queues, is also allowed.

The problem is to find a dynamic control policy which maximizes the concave utility function $H(x^*)$ (where x^* is the average value of $b(k(t))$), subject to the finite number of convex constraints (44), and *subject to the additional condition that the queues at the processing nodes remain stable*, that is (informally speaking) the processes $Q_n(t)$, $t \ge 0$, for all $n \in \mathcal{N}^p$ remain bounded.

Let us use notation

$$\bar{b}_n(k) \doteq \lambda_n(k) - \mu_n(k) + \sum_{\ell\in\mathcal{N}^p} \mu_\ell(k) p_{\ell n}(k),\ n \in \mathcal{N}^p,\ k \in K.$$

The meaning of $\bar{b}_n(k)$ is simple: this is the average increment $Q_n(t+1) - Q_n(t)$ of the queue length at a processing node n, that would be caused by control k at time t, *assuming that all processing node queues at time t are "large enough."*

We are now in position to define the (generalized) GPD algorithm.

GPD algorithm. *At time t choose a control*

$$k(t) \in \arg\max_{k\in K(m(t))} \left[\nabla H(X(t)) - \sum_{j\in\mathcal{J}} \beta Q_j(t) \nabla G_j(X(t))\right] \cdot b(k) - \sum_{n\in\mathcal{N}^p} \beta Q_n(t) \bar{b}_n(k)\,, \tag{55}$$

where $\beta > 0$ *is a (small) parameter,* $X(t)$ *is updated as in (50) and "virtual queue lengths"* $Q_j(t)$, $j \in \mathcal{J}$, *are updated as in (51). The initial values* $X(0) \in \tilde{V}$ *and* $Q_j(0) \ge 0$, $j \in \mathcal{J}$, *are fixed arbitrarily.*

Obviously, this algorithm is a generalization of the algorithm defined in Section 6.4, in that it applies to a more general model. The GPD algorithm (55) is also more general than that defined in [15], in that it applies to a more general *problem*, allowing additional—possibly non-linear—constraints (44).

The asymptotic optimality of the GPD algorithm (55), *under the appropriate non-degeneracy condition, which combines (48) and the non-degeneracy condition in [15]*, is proved analogously to the way it is done for its special case (49). The rate region is defined exactly the same way as for the model in [15]; and the analog of Theorem 3 is proved, again, analogously to the proof of Theorem 3 in [15]. The key circumstance here is that, although algorithm (55) applies to a more general model, *the corresponding underlying optimization problem and the dynamic system arising in the asymptotic limit under the GPD algorithm (55), are still within the framework of our Section 4,* with $\mathcal{N} = \mathcal{N}^u \cup \mathcal{N}^p$. (As in [15], the stability constraint for each queue $n \in \mathcal{N}^p$ "translates" into a linear constraint in the underlying optimization problem, and rescaled Q_n "becomes" the dual variable corresponding to this constraint.) Thus, the analog of Theorem 3 for the GPD algorithm (55) along with Theorem 1 (exactly as it is in Section 4) establish asymptotic optimality of the algorithm.

7 Examples of Section 3: Solutions

7.1 Example 1

Base station maintains the average rate estimate X_n per each user $n \in \mathcal{N}$, and the single virtual queue length Q, corresponding to constraint (4). Specialization of the GPD algorithm to the problem (3)–(4) is such that the scheduling decision $k(t)$ in slot t is chosen according to the following rule:

$$k(t) \in \underset{k \in K(m(t))}{\arg\max} -w(k) + \beta Q(t) \sum_{n \in \mathcal{N}} H_n'(X_n(t)) b_n(k), \tag{56}$$

and the Q_n and X_n's are updated as

$$X_n(t+1) = \beta b_n(k(t)) + (1-\beta) X_n(t), \tag{57}$$

$$Q_n(t+1) = \Big[Q(t) - \sum_{n \in \mathcal{N}} [H_n(X_n(t)) + H_n'(X_n(t)) \times (b_n(k(t)) - X_n(t))] + h^{\min} \Big]^+, \tag{58}$$

where $\beta > 0$ is a small parameter.

Our general results in Section 6 show that this algorithm is close to optimal if β is small. Note that the base station schedules wireless transmissions dynamically, based only on the current state of the radio channel and a small number of variables, updated according to very simple rules; it does *not* need to know the stationary distribution of the channel state, and it only needs to know the set of scheduling decision available in the current state of the channel. Thus, the above algorithm "solves" the underlying optimization problem (5)–(6) without the explicit knowledge of the region V.

7.2 Example 2

7.2.1 Formal solution

Let us first describe the special case of the GPD algorithm, which (asymptotically) solves the problem (7)–(8) of Section 3.2, assuming that all the variables can be updated in every time slot and are globally known. There are two variables—average rate estimate X_n and virtual queue length Q_n—corresponding to each traffic source n. Then the algorithm is such that the scheduling decision $k_i(t)$ by access point i in slot t is chosen according to the following rule:

$$k_i(t) \in \underset{k_i \in K_i(m_i(t))}{\arg\max} \sum_{n \in \mathcal{N}_i} b_n(k_i) \Big[H_n'(X_n(t)) - \beta \sum_{\ell \in \mathcal{R}(n)} Q^{(\ell)}(t) C_\ell'(X^{(\ell)}(t)) \Big], \tag{59}$$

and X_n and Q_n are updated as

$$X_n(t+1) = \beta b_n(t) + (1-\beta) X_n(t), \tag{60}$$

$$Q_n(t+1) = \Big[Q_n(t) + \sum_{\ell \in \mathcal{R}(n)} [C_\ell(X^{(\ell)}(t)) + C_\ell'(X^{(\ell)}(t)) \times (b^{(\ell)}(t) - X^{(\ell)}(t))] - d_n \Big]^+, \tag{61}$$

where, to simplify notation, we write $b_n(t)$ to mean $b_n(k_i(t))$ with the appropriate i (such that $n \in \mathcal{N}_i$), and denote

$$b^{(\ell)}(t) \doteq \sum_{s \in \mathcal{R}^{-1}(\ell)} b_s(t),$$

$$Q^{(\ell)}(t) \doteq \sum_{s \in \mathcal{R}^{-1}(\ell)} Q_s(t),$$

$$X^{(\ell)}(t) \doteq \sum_{s \in \mathcal{R}^{-1}(\ell)} X_s(t).$$

Note that (60) implies

$$X^{(\ell)}(t+1) = \beta b^{(\ell)}(t) + (1-\beta) X^{(\ell)}(t). \tag{62}$$

Remark. We ignored the fact that each utility function $C_\ell(z)$ is defined only for $0 \le z < c_\ell$, and not for all real z. This "difficulty" is not essential. First, given the setting of this example, $C_\ell(z)$ does not need to be defined for negative z at all. Second, we can always replace C_ℓ with a continuously differentiable function coinciding with it in $[0, z_*]$ and say linear in $[z_*, \infty)$. If we choose $z_* < c_\ell$ to be sufficiently close to c_ℓ, so that $C_\ell(z_*)$ is large enough, the optimal solutions of the original and modified problems are the same.

7.2.2 Distributed implementation

The idea of a "distributed" implementation of the algorithm is suggested by the form of expressions (59)–(62). Let us rewrite them as follows:

$$k_i(t) \in \underset{k_i \in K_i(m_i(t))}{\arg\max} \sum_{n \in \mathcal{N}_i} b_n(k_i)[H_n'(X_n(t)) - \beta W_{1,n}(t)], \quad (63)$$

$$X_n(t+1) = \beta b_n(t) + (1-\beta) X_n(t), \quad (64)$$

$$Q_n(t+1) = \left[Q_n(t) + W_{2,n}(t) - d_n\right]^+, \quad (65)$$

$$W_{1,n}(t) \doteq \sum_{\ell \in \mathcal{R}(n)} W_1^{(\ell)}(t), \quad (66)$$

$$W_{2,n}(t) \doteq \sum_{\ell \in \mathcal{R}(n)} W_2^{(\ell)}(t), \quad (67)$$

$$X^{(\ell)}(t+1) = \beta b^{(\ell)}(t) + (1-\beta) X^{(\ell)}(t), \quad (68)$$

$$W_1^{(\ell)}(t) \doteq Q^{(\ell)}(t) C_\ell'\left(X^{(\ell)}(t)\right), \quad (69)$$

$$W_2^{(\ell)}(t) \doteq C_\ell\big(X^{(\ell)}(t)\big) + C_\ell'\big(X^{(\ell)}(t)\big)\big(b^{(\ell)}(t) - X^{(\ell)}(t)\big). \quad (70)$$

Then a distributed algorithm is as follows. Each access point i "maintains" variables X_n, Q_n, $W_{1,n}$ and $W_{2,n}$ for "its" sources n. Each link ℓ maintains variables $X^{(\ell)}$, $b^{(\ell)}$ and $Q^{(\ell)}$, which in turn determine $W_1^{(\ell)}$ and $W_2^{(\ell)}$. Clearly, for each source n, the access point can update X_n by simply observing traffic it transmits. However, the updates of Q_n and $W_{1,n}$ depend on the values of $W_1^{(\ell)}$ and $W_2^{(\ell)}$ on the links $\ell \in \mathcal{R}(n)$ along the route of flow n. Similarly, while each link ℓ can update $X^{(\ell)}$ and $b^{(\ell)}$ by simply observing the *aggregate* amount of traffic passing through the link, the updates of $Q^{(\ell)}$ require the knowledge of Q_n for the flows feeding this link. A "light-weight" mechanism for such an information exchange between traffic sources and links can be, for example, as follows. Periodically, say every τ slots, $\tau \geq 1$, the access point i sends a small high priority ("signaling") message along the route of each of its flows n. The message along flow n route contains the current value of Q_n, and also "data fields" $W_{1,n}$ and $W_{2,n}$ initialized to 0. When a signaling message of flow n passes through link ℓ, the link, first, uses the value of Q_n to update its $Q^{(\ell)}$ and, second, increments fields $W_{1,n}$ and $W_{2,n}$ by the current values of $W_1^{(\ell)}$ and $W_2^{(\ell)}$, respectively. When a signaling message reaches destination, it is sent back to the source. When an access point i receives back a signaling message for flow n, it takes the value of the field $W_{1,n}$ as the new value of this variable, and it uses the field $W_{2,n}$ to update Q_n.

We will not provide further specifics or describe numerous possible modifications of the mechanism described above, because we believe they can be filled in by an interested reader. We only note that the delays in updating variables, depending in particular on the "update frequency" $1/\tau$, *do not "destroy" the asymptotic optimality of the algorithm, as long as such delays are bounded*—it is easy to see that the dynamic system obtained by rescaling and taking the limit as $\beta \to 0$ is exactly the same as the dynamic system under the original (delay free) algorithm (63)–(70).

We do want to emphasize the "distributed nature" of the above algorithm. Indeed, traffic sources and access points need not know the form of congestion cost functions along the flow routes, or in fact the exact routes themselves. Access points schedule wireless transmissions dynamically, based on the current state of the radio channel to its users (along with their corresponding variables X_n, $W_{1,n}$, and utility functions H_n). Wireline network links do *not* need to know the utility functions of the flows, or congestion cost functions on the other links. In fact, if the network design can assure that, within each window of τ slots, each link receives exactly one signaling message per each flow passing through the link, the link does not need to maintain the list of flows it currently serves; otherwise, the link can maintain some form of such a list.

Appendix: Auxiliary elementary facts

The following Proposition 1 is Lemma 20 in [15]. Note that in the differential inclusion (71) it is only required that $v(t) \in V$. (There are *no* other conditions on $v(t)$.)

Proposition 1. *Suppose V is a convex compact subset of a finite-dimensional space R^N. Suppose a vector-function $(x(t),\ t \geq 0)$, taking values in R^N, is absolutely continuous, satisfying the following differential inclusion for almost all $t \geq 0$:*

$$x'(t) = v(t) - x(t), \qquad v(t) \in V\ . \quad (71)$$

Then

(i) *Both $x(t)$ and the distance $\rho(x(t), V)$ are Lipschitz continuous in $[0, \infty)$;*

(ii) *The distance* $\rho(x(t), V)$ *is non-increasing, and moreover, for almost all* $t \geq 0$,

$$\frac{d}{dt}\rho(x(t), V) \leq -\rho(x(t), V) , \tag{72}$$

which implies that

$$\rho(x(t), V) \leq \rho(x(0), V)e^{-t} .$$

(iii) *The entire trajectory* $(x(t),\ t \geq 0)$ *is contained within the convex hull of* $V \cup \{x(0)\}$.

The following elementary fact is the Proposition 1 of [15].

Proposition 2. *Suppose* $H(v)$ *is a continuously differentiable concave function with convex open domain* $\tilde{V} \subseteq R^N$, $N \geq 1$, *and let* L *be the line segment connecting two points* $v^{(1)}, v^{(2)} \in \tilde{V}$. *If* $H(v)$ *has the same value along segment* L, *then* $\nabla H(v^{(1)}) = \nabla H(v^{(2)})$. (*Consequently, if* $H(v)$ *is linear along* L, *then* $\nabla H(v)$ *is the same for all* $v \in L$).

References

1. M. Andrews, K. Kumaran, K. Ramanan, A.L. Stolyar, R. Vijayakumar, and P. Whiting, Providing quality of service over a shared wireless link. IEEE Communications Magazine 39(2) (2001) 150–154.
2. K.J. Arrow, L. Hurwicz, and H. Uzawa (eds.), *Studies in Linear and Non-Linear Programming* (Stanford University Press, 1958), Chapters 6–8.
3. D.P. Bertsekas, *Constrained Optimization and Lagrange Multiplier Methods* (Academic Press, 1982).
4. P. Bender, P. Black, M. Grob, R. Padovani, N. Sindhushayana, and A. Viterbi, CDMA/HDR: A bandwidth efficient high speed wireless data service for nomadic users. IEEE Communications Magazine (July 2000).
5. D. Boyd and L. Vandenberghe, *Convex Optimization* (Cambridge University Press, 2004).
6. S.N. Ethier and T.G. Kurtz, *Markov Process: Characterization and Convergence* (John Wiley and Sons, New York, 1986).
7. A. Eryilmaz and R. Srikant, Fair resource allocation in wireless networks using queue-length-based scheduling and congestion control. In *Proceeding of INFOCOM'2005*, Miami (March 13–17 2005).
8. F.P. Kelly, A.K. Maullo, and D.K.H. Tan, Rate control in communication networks: Shadow prices, proportional fairness and stability. Journal of the Operational Research Society 49 (1998) 237–252.
9. F.P. Kelly, Fairness and stability of end-to-end congestion control. European Journal of Control 9 (2003) 159–176.
10. X. Lin and N.B. Shroff, The impact of imperfect scheduling on cross-layer rate control in wireless networks. In *Proceeding of INFOCOM'2005*, Miami (March 13–17 2005).
11. S. Low, A duality model of TCP and queue management algorithms. IEEE/ACM Transactions on Networking 11(4) (2003) 525–536.
12. M. Neely, E. Modiano, and C. Li, Fairness and optimal stochastic control for heterogeneous networks. In *Proceeding of INFOCOM'2005*, Miami (March 13–17 2005).
13. A. L. Stolyar, MaxWeight scheduling in a generalized switch: State space collapse and workload minimization in heavy traffic. Annals of Applied Probability 14(1) (2004) 1–53.
14. A.L. Stolyar, On the asymptotic optimality of the gradient scheduling algorithm for multi-user throughput allocation. Operations Research 53(1) (2005) 12–25.
15. A.L. Stolyar, Maximizing queueing network utility subject to stability: Greedy primal-dual algorithm. Queueing Syst. 50 (2005) 401–457.
16. P. Viswanath, D. Tse and R. Laroia, Opportunistic beamforming using dumb antennas. IEEE Transactions on Information Theory 48(6) (2002) 1277–1294.