Stability of Global LIFO Networks

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Abstract

It is proved that a general open multi-class queueing network with the Global preempt-resume Last-In-First-Out (G-LIFO) multi-channel discipline is stable under the standard sub-criticality condition $\rho_j < r_j$. Here r_j is the number of service channels and $\rho_j = \sum_{(i,k)\in\mathcal{G}_j} \lambda_i v_{ik}$ the 'nominal' load in node j; λ_i is the exogenous rate of arrival of customers of class i and v_{ik} the mean service time of the class i customers at the kth node on the route. $\mathcal{G}_j$ is the set of types (i,k) of the customers served in node j. This result contrasts to examples of multi-class FIFO queueing networks where the nominal sub-criticality condition does not guarantee stability.

1 The G-LIFO service discipline

Consider an open queueing network with several customer classes. The class of a customer determines its route through the network and the distribution of the service time in each node on the route. At the time of arrival from outside a customer enters the first node of its route. After finishing service in a given node, a customer instantly enters the next node of its route; if it was the last node of the route, the customer leaves the network. The queueing discipline adopted in this paper is multi-channel Global preempt-resume Last-In-First-Out (G-LIFO). This means that customers in each node are served in the inverse order of their *exogenous* arrival times (i.e., times of arrival *in the network*); a new customer interrupts the current service if the customer under service is 'older'. More precisely, suppose that a new customer, say A, with an exogenous arrival time t, enters a given node (from outside or after being served in the previous node of its route). If there is an idle channel in the node, the newcomer takes it. Otherwise, two cases can occur: a) all customers under service are 'younger' than A, i.e., have exogenous times $\geq t$, and b) the 'oldest' customer under service has an exogenous time $t' < t$; call it A$'$. In case a) customer A waits until the time when (i) one of the channels completes service and (ii) all other waiting customers (regardless of whether their service was interrupted or not) have their exogenous times $< t$. Then A occupies the available channel. In case b) A interrupts the service of customer A$'$ and takes over the corresponding channel. The interrupted service is then resumed at the first time when a channel in the node completes service and all other waiting customers have their exogenous times $< t$. All ties are broken at random (in the case of continuously distributed random variables they occur with probability 0).

So, between points of exogenous arrival and end of service the network functions 'smoothly': customers under service diminish their residual service times at rate one, while the age (i.e., the time lapsed from the time of exogenous arrival) of all customers present in the network (i.e., served or waiting in the network nodes) increases at rate one. We show that if the nominal load at each node in the network is strictly less than the number of channels then the network is stable. By stability we mean ergodicity (more precisely, positive Harris recurrence) of the underlying stochastic process.

This fact is in contrast with the well-known results (see [2], [3]) showing that in the case of the FIFO (First-In-First-Out) discipline, the nominal sub-criticality condition is not enough for stability of a multi-class network.

It has to be said that the G-LIFO discipline has its advantages (it is simple to implement, distributes the workload evenly between the nodes of the route) and disadvantages (creates a backlog of interrupted work).

It is easy to check that if the nominal sub-criticality condition is reversed in at least one node $j \in \mathcal{J}$ (in the sense that the nominal load is strictly greater than the number of channels in the node) then the network is unstable.

The proofs given in this paper are based on the method of the *fluid limit* [12, 5, 4, 13, 6], which reduces the problem of verifying of a *stochastic* network to verifying stability of a corresponding *deterministic* fluid network.

The argument we use to show stability of the fluid network follows closely the one used by Andrews et al. [1] to show stability of the network with *Shortest-in-System* discipline. (They consider a network with possibly *adversarial* behavior of input flows with the constraints on the input rates. The stability is understood as the property that the number of customers in the network remains bounded as time goes to infinity.) However, the "translation" of the argument into the framework of a stochastic network is not straightworward, and this is where the main contribution of this paper is.

2 The arrival and service times

We consider a network with a finite set of nodes $\mathcal{J} = \{1, 2, \ldots, J\}$. Each network node j is a multi-channel queue, with r_j service channels of rate one. There are finitely many different customer classes forming the set $\mathcal{I} = \{1, 2, \ldots, I\}$. The exogenous arrival flow of customers of each class $i \in \mathcal{I}$ is such that the inter-arrival times are i.i.d. random variables $\xi_i(1)$, $\xi_i(2)$, …, with densities and a finite non-zero mean equal to $1/\lambda_i$. This means that λ_i is the exogenous arrival rate for class i.

A class i customer has its prescribed route through the network,

$$\hat{\jmath}(i,1),\ \ldots,\ \hat{\jmath}(i,k),\ \ldots, \hat{\jmath}(i,K(i)),$$

where $K(i)$ is the length and $\hat{\jmath}(i,k) \in \mathcal{J}$ the kth node of the route. After completing service in node $\hat{\jmath}(i,k)$ the customer enters node $\hat{\jmath}(i,k+1)$ or – if $k = K(i)$ – leaves the network. A class i customer in k-th node of his route will be called a *type* (i,k) *customer*, or an (i,k)-*customer*.

All service times of the customers are mutually independent and independent of their exogenous arrival times and the times they enter the nodes of their routes. Given $i \in \mathcal{I}$ and $k \in \{1, .., K(i)\}$, the service times of (i,k)-customers are i.i.d., with mean $v_{ik} > 0$.

Denote by $\mathcal{G}$ the whole set of customers' types and by $\mathcal{G}_j$ the subset of $\mathcal{G}$ listing the types of the customers to be served in node $j \in \mathcal{J}$:

$$\mathcal{G} = \{(i,k) \mid k = 1, 2, \ldots, K(i);\ \ i \in \mathcal{I}\},\ \mathcal{G}_j = \{(i,k) \in \mathcal{G} \mid \hat{\jmath}(i,k) = j\}.$$

We suppose that $\forall\, j \in \mathcal{J}$ the nominal load in node j is less than r_j, i.e.

$$\rho_j \equiv \sum_{(i,k)\in\mathcal{G}_j} \lambda_i v_{ik} < r_j. \tag{1}$$

3 Stability. Preliminaries.

A state of the network is

$$X = \{((V_{i,k,m}, A_{i,k,m};\ m = 1, \ldots, M_{i,k}),\ (i,k) \in \mathcal{G}),\ (U_i,\ i \in \mathcal{I})\}.$$

Here (a) $M_{i,k}$ is the number of the (i,k)-customers in the network, (b) $V_{i,k,m}$ is the residual service time and $A_{i,k,m}$ the age of the m-th (i,k)-customer (for each type (i,k) we list the (i,k)-customers in the ascending order of their age), (c) U_i is the residual inter-arrival time for flow i. The number of customers in state X present in node $j \in \mathcal{J}$ equals $N_j(t) = \sum_{(i,k):\, \hat{\jmath}(i,k)=j} M_{i,k}$; we assume that $\min[r_j, N_j(t)]$ 'youngest' of them are under service and the rest are waiting. The evolution of a state is then defined by the above description (see Section 1); it is clear that $X(t), t \geq 0$, is a Markov process.

The norm of state X is defined as

$$\|X\| = \sum_i U_i + \sum_{(i,k)\in\mathcal{G}} M_{i,k} + \sum_{(i,k)\in\mathcal{G}} \sum_{m=1}^{M_{i,k}} (V_{i,k,m} + A_{i,k,m}).$$

We also impose the following technical conditions on the distribution of the inter-arrival time $\xi_i(1)$, (cf. condition (25) in [9]). For each $i \in \mathcal{I}$, there exist an integer $l \geq 1$ and a function $p(y) \geq 0, y \geq 0$, with $\int_0^\infty p(y)dy > 0$ such that

$$\Pr\{\xi_i(1) \geq y\} > 0 \quad \text{for any} \quad y > 0\ , \tag{2}$$

$$\Pr\{c_1 \leq \sum_{n=1}^{l} \xi_i(n) \leq c_2\} \geq \int_{c_1}^{c_2} p(y)dy \quad \text{for any} \quad 0 \leq c_1 \leq c_2. \tag{3}$$

Conditions (1)–(3) are assumed throughout the paper.

The stability of a random process is understood as *positive Harris recurrence* (see, for example, [10] or [5] for the exact definition.)

To prove stability of process $X(t)$, we will use the "fluid limit" approach introduced in [12] and further developed in [5], [4], [13], [6]. This approach is based on the following result.

Theorem 1 *Suppose that there exists a constant $T > 0$ such that for any sequence of processes $X^{(a_n)}(t)$, with initial states $X^{(a_n)}(0)$ of norm $\|X^{(a_n)}(0)\| = a_n \to \infty$ $(n \to \infty)$, the expected value of $\frac{1}{a_n}\|X^{(a_n)}(a_n T)\|$ approaches 0:*

$$\lim_{n\to\infty} E[\frac{1}{a_n}\|X^{(a_n)}(a_n T)\| = 0 \tag{4}$$

Then the Markov process X is positive Harris recurrent.

Theorem 1 for continuous-time countable Markov chains was proved by Rybko and Stolyar [12] and generalized to the form presented above by Dai [5], following results of [10] (see also [11]). For discrete-time countable Markov chains, Theorem 1 is a special case of a more general result established by Malyshev and Menshikov [8] (see also [7]).

The fluid limit approach works as follows. As the form of the condition (4) suggests, one needs to consider a "fluid process" $x(t)$ obtained as a limit of the sequence of scaled processes $\frac{1}{a_n}X^{(a_n)}(a_n t), a_n \to \infty$, and show that $x(t)$ starting from any initial state with norm $\|x(0)\| = 1$ reaches 0 by some fixed finite time T and stays there. As a rule, the latter property implies (4) (see [5], [13]).

In the definitions that follow, customers' arrival always means their exogenous arrival. Also, counting customers arrived by a time t, we include, when appropriate, customers present at time 0. Consider random processes $F_i^{(a_n)}(t)$, $W_{i,k}^{(a_n)}(t)$, $\widehat{F}_{i,k}^{(a_n)}(t_1, t_2)$, and $\widehat{W}_{i,k}^{(a_n)}(t_1, t_2)$ associated with process $X^{(a_n)}(t)$. 1) $F_i^{(a_n)}(t)$, $t \geq -a_n$, is the number of the i-customers arrived prior time t (as we consider process $X(t)$ starting at $t = 0$ from a general non-empty state, some customers have arrived before time 0 but not before time $-a_n$; see the definition of norm $\|X\|$). 2) IMPORTANT CLARIFICATION HERE $W_{i,k}^{(a_n)}(t)$, $t \geq 0$, is the total amount of work (i.e., total required service time) *intended for* the node $\hat{\jmath}(i,k)$, due to the i-customers arrived into the *network* by time t, including customers present at time 0. 3) $\widehat{F}_{i,k}^{(a_n)}(t_1, t_2)$, $t_1 \geq -a_n$, $t_2 \geq 0$, $t_1 < t_2$, is the total number of (i,k)-customers arrived by time t_1 and completely served in node $\hat{\jmath}(i,k)$ by time t_2. 4) $\widehat{W}_{i,k}^{(a_n)}(t_1, t_2)$, $t_1, t_2 \geq 0$, $t_1 < t_2$, is the amount of time spent by the channels in node $\hat{\jmath}(i,k)$ by time t_2 while serving the (i,k)-customers arrived before time t_1. All these processes (and processes that are introduced below) are assumed to be right-continuous with respect to t, t_1 and t_2.

4 Fluid Limit Properties

Define the *scaled* processes as follows

$$f_i^{(a_n)}(\tau) = \frac{1}{a_n} F_i^{(a_n)}(a_n \tau), \quad \tau \geq -1,$$

$$w_{i,k}^{(a_n)}(\tau) = \frac{1}{a_n} W_{i,k}^{(a_n)}(a_n \tau), \quad \tau \geq 0,$$

$$\hat{f}_{i,k}^{(a_n)}(\tau_1, \tau_2) = \frac{1}{a_n} \widehat{F}_{i,k}^{(a_n)}(a_n \tau_1, a_n \tau_2), \quad \tau_1 \geq -1,\ \tau_2 \geq 0,\ \tau_1 < \tau_2,$$

$$\hat{w}_{i,k}^{(a_n)}(\tau_1, \tau_2) = \frac{1}{a_n} \widehat{W}_{i,k}^{(a_n)}(a_n \tau_1, a_n \tau_2), \quad \tau_1 \geq -1,\ \tau_2 \geq 0,\ \tau_1 < \tau_2$$

Here, τ, τ_1 and τ_2 are rescaled times; the original processes $F_i^{(a_n)}(t)$, $W_{i,k}^{(a_n)}(t)$, $\widehat{F}_{i,k}^{(a_n)}(t_1, t_2)$ and $\widehat{W}_{i,k}^{(a_n)}(t_1, t_2)$ 'live' in the time that is a_n times 'faster'. We need one more process associated with $f_i^{(a_n)}(\tau)$ and $\widehat{f}_{i,k}^{(a_n)}(\tau_1,\tau_2)$. For τ_1 and τ_2 such that $-1 \le \tau_1 < \tau_2$ and $\tau_2 \ge 0$, define

$$g_{i,k}^{(a_n)}(\tau_1,\tau_2) = \inf\{\tau \ge \tau_2 \ : \ \widehat{f}_{i,k}^{(a_n)}(\tau,\tau) - \widehat{f}_{i,k}^{(a_n)}(\tau_1,\tau) = f_i^{(a_n)}(\tau) - f_i^{(a_n)}(\tau_1)\} \ .$$

In other words, $g_{i,k}^{(a_n)}(\tau_1,\tau_2)$ is an upper bound of the first time after τ_2 when all (i,k)-customers arrived in the network in the rescaled-time interval $(\tau_1,\tau_2]$ complete service by node $\hat{\jmath}(i,k)$.

The following lemma can be considered as a variant of Theorem 4.1 in [5] or Theorem 7.1 in [13].

Lemma 1 *Fix a sequence of processes $\{X^{(a_n)}(t)\}$ with initial states of norm $\|X^{(a_n)}(0)\| = a_n \to \infty$ and a constant $T_1 > 1$. Then with probability 1 any subsequence of $\{X^{(a_n)}(t)\}$ in turn contains a subsequence $\{X^{(a'_m)}(t),\ \{a'_m\} \subseteq \{a_n\}\}$ such that as $m \to \infty$, the following limits hold for every type $(i,k) \in \mathcal{G}$:*

(i)

$$f_i^{(a'_m)}(T_1) \to f_i(T_1), \tag{5}$$

$$w_{i,k}^{(a'_m)}(T_1) \to w_{i,k}(T_1), \tag{6}$$

where $f_i(T_1)$ and $w_{i,k}(T_1)$ are some non-negative constants.

(ii) *Uniformly in $\tau \ge T_1$ within a compact set,*

$$f_i^{(a'_m)}(\tau) - f_i^{(a'_m)}(T_1) \ \to \ \lambda_i(\tau - T_1), \tag{7}$$

$$w_{i,k}^{(a'_m)}(\tau) - w_{i,k}^{(a'_m)}(T_1) \ \to \ \lambda_i v_{i,k}(\tau - T_1), \tag{8}$$

$$\widehat{f}_{i,k}^{(a'_m)}(\tau,\tau) - \widehat{f}_{i,k}^{(a'_m)}(T_1,\tau) \ \to \ \lambda_i(\tau - T_1), \tag{9}$$

$$\widehat{w}_{i,k}^{(a'_m)}(\tau,\tau) - \widehat{w}_{i,k}^{(a'_m)}(T_1,\tau) \ \to \ \lambda_i v_{i,k}(\tau - T_1), \tag{10}$$

$$g_{i,k}^{(a'_m)}(T_1,\tau) \ \to \ \tau. \tag{11}$$

Proof : The choice of $T_1 > 1$ guarantees that all 'initial' rescaled inter-arrival times are over strictly before T_1. By the functional strong law of large numbers we have that with probability one, uniformly in $s \ge 0$ within a compact set, (i) for each $i \in \mathcal{I}$,

$$f_i^{(a_n)}(s + U_i^{(a_n)}(0)/a_n) - f_i^{(a_n)}(U_i^{(a_n)}(0)/a_n) \ \to \ \lambda_i s, \tag{12}$$

and (ii) for each $(i,k) \in \mathcal{G}$,

$$w_{i,k}^{(a_n)}(s + U_i^{(a_n)}(0)/a_n) - w_{i,k}^{(a_n)}(U_i^{(a_n)}(0)/a_n) \ \to \ \lambda_i v_{i,k} s. \tag{13}$$

Properties (5), (6), (7), and (8), follow from (12) and (13) if subsequence $\{a'_m\}$ is chosen such that $U_i^{(a'_m)}(0)/a'_m$ converges for each i.

Properties (9) and (10) follow from (11). The latter property is proved for each (i,k) by induction on $k = 1, 2, \ldots$. Below we show the proof for $(i,1)$ only (which establishes the initial stage of induction); the completion of the induction step is a mere repetition of the argument.

Fix class $i \in \mathcal{I}$ and consider the type $(i,1)$ flow served by node $j = \hat{j}(i,k)$. Fix $\epsilon > 0$ and τ with $T_1 \leq \tau - \epsilon < \tau$. We know from (8) that

$$\lim w_{i,1}^{(a'_m)}(\tau) - w_{i,1}^{(a'_m)}(\tau - \epsilon) = \lambda_i v_{i,1} \epsilon \ .$$

Let $\delta > 0$ be another constant. Consider an $(i,1)$-customer with a rescaled arrival time within $(\tau - \epsilon, \tau]$. The total workload able to "compete" for the service at node j with the above customer in the interval $(\tau, \tau + \delta]$, is at most

$$\sum_{(l,k) \in \mathcal{G}_j} w_{l,k}^{(a'_m)}(\tau + \delta) - w_{i,1}^{(a'_m)}(\tau - \epsilon) \to \rho_j(\epsilon + \delta) \ .$$

Choose $\delta = \delta(\epsilon) > (\rho_j / r_j)(\epsilon + \delta)$ so that $\delta(\epsilon) \to 0$ as $\epsilon \to 0$ (e.g., $\delta(\epsilon) = 2(\rho_j/(r_j - \rho_j))\epsilon$). Then our $(i,1)$-customer must be completely served strictly before rescaled time $t + \delta$.

Since any fixed interval $(T_1, t]$ can be broken down into a finite number of ϵ-length intervals, and since $\delta(\epsilon) \to 0$ as $\epsilon \to 0$, we obtain (11) for $(i,1)$, uniformly on compacts. This implies properties (9) and (10) for our particular i and $k = 1$.

The induction in k (conducted in a similar fashion) completes the proof of the lemma.

Lemma 2 *There exists a constant $T > 0$ such that for any $\tau \geq T$ and any sequence of processes $\{X^{(a_n)}(t)\}$ with initial states of norm $\|X^{(a_n)}(0)\| = a_n \to \infty$, with probability* 1,

$$\lim_{n \to \infty} \frac{1}{a_n} \|X^{(a_n)}(a_n \tau)\| = 0 \ . \tag{14}$$

Proof : Let $\{X^{(a_n)}(t)\}$ be a sequence of processes as in Lemma 1. It suffices to show the existence of a (uniform, independent of the subsequence) constant $T > 0$ such that for any $\tau \geq T$, with probability one any subsequence of $\{X^{(a_n)}(t)\}$ contains another subsequence $\{X^{(a'_m)}(t)\}$ such that (14) holds.

Fix $T_1 > 0$. With probability one, any subsequence contains another one, $\{X^{(a'_m)}(t)\}$, such that the assertion of Lemma 1 holds. Hence, in any rescaled-time interval $(\tau_1, \tau_2] \subset (T_1, \infty]$ the total rescaled time that the channels in node j spend while serving customers arrived in the network after T_1 is asymptotically equal to $\rho_j(\tau_2 - \tau_1)$. This means that the remaining time $(r_j - \rho_j)(\tau_2 - \tau_1)$ is available for servicing the workload arrived at or before T_1. This implies the existence of $T > 0$ such that for all sufficiently large values of scaling parameter a'_m, all the customers arrived before time T_1 will leave the network by time T.

It remains to observe that Lemma 1 implies that for any time $t > T_1$, the contribution into the scaled norm $\|X^{(a'_m)}(a'_m t)\| / a'_m$ of the customers arrived after rescaled time T_1 vanishes as $a'_m \to \infty$.

5 Main Result

Lemma 2 and the uniform integrability of the family of random variables $\|X^{(a'_m)}(a'_m T)\|/a'_m$ verify the condition in Theorem 1. (The uniform integrability is easily established by majorizing the state norm at time t by that of a "worst case scenario". For example, the numbers of customers are replaced by the total number of customers arrived by time t, and the residual service times are replaced by the total service time of all customers arrived by time t. Cf. [12, 5].) Thus we have proved the following

Theorem 2 *Under conditions* (1)–(3)*, Markov process* $X(t)$*,* $t \geq 0$*, for the network with the G-LIFO discipline is positive Harris recurrent.*

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