PROBLEMS OF INFORMATION TRANSMISSION

A translation of *Problemy Peredachi Informatsii*

January, 1993

Volume 28, Number 3 **July-September, 1992**

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The Russian press date (podpisano k pechati) of this issue was 7/30/1992. Publication therefore did not occur prior to this date, but must be assumed to have taken place reasonably soon thereafter.

ERGODICITY OF STOCHASTIC PROCESSES DESCRIBING THE OPERATION OF OPEN QUEUEING NETWORKS

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UDC 621.394.74:519.2

We consider stochastic processes that model the operation of open queueing networks with calls of different types. Each call type has its own route. A new ergodicity criterion is proposed for continuous-time countable Markov chains. This criterion is applied to reduce the problem of finding the ergodicity conditions of the Markov process that describes the operation of the queueing network to the analysis of the behavior of a special (limiting) deterministic process obtained from the original process by normalization and a change of time scale. For the simplest nontrivial network of this class – a two-node network with two types of calls moving in opposite directions, the natural condition of "less than unit load at each node" is sufficient for ergodicity of the modeling Markov process under the FCFS discipline. An example of a simple priority discipline is considered for which the corresponding Markov process is nonrecurrent under this condition.

1. INTRODUCTION

We study the existence conditions of a stationary operating mode in open queueing networks with finitely many calls. Calls of each type have a fixed path through the network. The mean service time at the network nodes depends on the type of call and the serial number of the node along the path of the call. We consider only networks whose operation is described by a continuous-time homogeneous Markov process with countably many states. This restriction is technical: it is imposed because the ergodicity criterion is derived specifically for such processes.

The queueing network consists of J nodes and there are I types of calls in the network. The input stream of calls of type i, $i \in \{1,\ldots,I\}$, is a Poisson process with rate λ_i (these processes are mutually independent for calls of different types). Each call type has its own path through the network

$$\hat{j}(i, 1), \ldots, \hat{j}(i, k), \ldots, \hat{j}(i, K(i)),\ \hat{j}(i, k) \in J,\ k = 1, \ldots, K(i),$$

i.e., a sequence of network nodes that serve calls of the given type before they leave the network. Here $K(i)$ is the path length for calls of type i and $\hat{j}(i, k) \in J (k = 1,\ldots,K(i))$ is the network node where a call of type i is served in step k along its path.

Each node consists of a single server and a queue with an unlimited number of waiting places. The service times of all calls in all nodes are independent random variables. By ν_{ik}, $i \in I$, $k = \{1,\ldots,K(i)\}$, we denote the mean service time of a call of type i in step k of the path (i.e., in node $\hat{j}(i, k)$). Let $\nu_{ik} > 0$, $\forall i \in I$, $\forall k \le K(i)$. Assume for simplicity that all service times are exponentially distributed (all our results remain valid for an arbitrary phase distribution of service times). Some conservative queueing discipline acts in each node $j \in J$.

It is easy to determine the homogeneous Markov process $s(t)$ with a countable phase space that describes the operation of this network. Processes of this kind have been previously studied by Rybko [1, 2], Kelly [3], and Massey [4].

Denote by ρ_j the total load of node j:

$$\rho_j = \sum_{(i,k):\ \hat{j}(i,k) = j} \lambda_i \nu_{ik}.$$

The following simple theorem is proved in [1].

Translated from Problemy Peredachi Informatsii, Vol. 28, No. 3, pp. 3-26, July-September, 1992. Original article submitted June 17, 1991.

0032-9460/92/2803-0199$12.50

THEOREM. If $s(t)$ is an ergodic Markov process, then for all $j \in J$,

$$\rho_j < 1.$$

Dobrushin has suggested the following natural conjecture.

Conjecture 1. If for all $j \in J$

$$\rho_j < 1,$$

then $s(t)$ is an ergodic Markov process.

Kelly [5] and Massey [4] have found many nontrivial classes of networks for which the stationary distribution of the process $s(t)$ has a product form (and conjecture 1 is therefore true). However, the product-form condition imposes restrictive requirements on the parameter v_{ik} and on the queueing discipline and does not apply in many natural cases, e.g., in case of a "first come, first served" (FCFS) discipline in each node.

Conjecture 1 also has been proved for two simple classes of networks (see [2]). Our approach relies on the analysis of some "limit" deterministic process (see Sec. 4). This deterministic process may be viewed as the limit of a sequence of processes that are obtained by normalization and change of time scale from the sequence of original processes whose initial state "goes to infinity." This technique is also useful for studying the existence of the stationary operating mode for more general networks.

We start with an illustrative and not quite formalized interpretation of this limit deterministic process. Given are J vessels with I types of liquids. Initially, the total quantity of liquids in all vessels equals 1. Liquid of type $i \in I$ is poured into the first vessel $\hat{j}(i, 1)$ along its path at a constant rate λ_i, flows through the sequence of vessels

$$\hat{j}(i, 1), \ldots, \hat{j}(i, k), \ldots, \hat{j}(i, K(i))$$

and then leaves the network. The "volume" of one unit of liquid i in vessel $j = \hat{j}(i, k)$ is v_{ik}. Each vessel $j \in J$ has two openings: the top opening through which liquid is admitted into the vessel (the size of this opening is unrestricted) and the bottom opening through which the liquid flows out of the vessel. The size of the bottom opening is 1. Thus, if a particular vessel is not empty, then the total "volume" of liquid leaving this vessel in unit time equals 1. Liquid flows out of the bottom opening in each vessel according to the same discipline as the queueing discipline for the corresponding node in the queueing network. For instance, with FCFS discipline, the liquids flow out of the vessel "in the same order" in which they are poured into the vessel. The ergodicity property of the corresponding Markov process indicates that our system of vessels is emptied in a finite time starting from any initial state with unit total quantity of liquids. Such discrete deterministic processes are studied in scheduling theory of manufacturing systems (see Perkins and Kumar [6], Kumar and Seidman [7]). The example of an unstable deterministic process in Theorem 6 is similar to the Kumar and Seidman example in [7]. This example helps to prove nonrecurrence of the corresponding Markov process (Theorem 7).

In this paper, we consider the simplest nontrivial network (as described in Sec. 3) which cannot be analyzed by the previously developed method. This network consists of two nodes that serve two types of calls. The calls move through the network in opposite directions. Specifically, type-1 calls arriving in the network are first served in node 1 and then in node 2. Type-2 calls conversely are first served in node 2 and then in node 1. Calls of both types are enqueued in the same queue at each node.

The first case analyzed in this paper is a network with FCFS discipline in both nodes. Conjecture 1 is proved for this case (Theorem 1). Theorem 1 is proved by a Moustafa–Foster type test for continuous-time homogeneous Markov processes with countably many states (Theorem 2). A similar (more general) ergodicity test for discrete-time countable Markov chains has been proved by Malyshev and Men'shikov [8].

The second case considered in this paper involves special priority service in each node: type-2 calls have absolute priority in node 1 and conversely type-1 calls have absolute priority in node 2. We show (Theorem 7) that conjecture 1 is not true in this case: for some region of parameters satisfying the condition

$$\rho_j < 1, j \in J,$$

the corresponding Markov process is nonrecurrent.

The paper is organized as follows. A formal description of the two-node network is given in Sec. 2. Conjecture 1 is proved in Secs. 3-5 for the case of a network with FCFS discipline in both nodes. Section 6 considers the priority discipline and proves nonrecurrence of the corresponding Markov process for some region of the parameters ν_{ik} that satisfy the condition of conjecture 1. The proofs of Lemmas 1, 2, 3, 1′, 2′, and 4 and Theorem 2 are given in the Appendix.

2. FORMAL DESCRIPTION OF THE MODEL

The network has two nodes $J = \{1, 2\}$. Each node consists of a server and a queue with an unlimited number of waiting places. Two types of calls arrive in the network ($I = \{1, 2\}$). The call arrivals are independent Poisson processes with rates λ_i, $i \in I$. Fixed paths are associated to calls of each type: type-1 calls are first served in node 1 and then in node 2; type-2 calls conversely are first served in node 2 and then in node 1. Each call leaves the network when its service is completed in the second node along its path. We say that type-i calls reaching the k-th node along their path (i.e., the node $j = \hat{j}(i, k)$) form an (i, k)-stream, $i \in I$, $k = 1, 2$. For instance, $\hat{j}(2, 1) = 2$, the (2, 1)-stream is a Poisson process with rate λ_2. Recall that the service times are independent exponential random variables with mean ν_{ik}, $i \in I$, $k = 1, 2$, for (i, k)-stream calls (in node $j = \hat{j}(i, k)$).

In what follows we assume that

$$\rho_j < 1, j = 1, 2. \tag{1}$$

3. NETWORK WITH FCFS DISCIPLINE

Let the FCFS discipline be specified in each node. The network state $s \in S$ is defined as

$$s = \{(s_{jl}, l = 1, \ldots, Q_j, j \in J\}.$$

Here Q_j is the total number of calls in node $j \in J$, $s_{jl} \in G_j \triangleq \{(i, k)\colon \hat{j}(i, k) = j\}$ is the type of the call enqueued in place l at node j. Clearly, $s(t)$, $t \geq 0$, is an irreducible countable Markov chain in continuous time t.

THEOREM 1. When condition (1) is satisfied, the Markov process $s(t)$ is ergodic.

To prove the theorem, we need the following ergodicity test for continuous-time homogeneous Markov processes with countably many states (it is an analog of the ergodicity test derived in [8] for discrete-time countable Markov chains).

THEOREM 2. Consider an irreducible Markov chain with a countable state set S and continuous time $t \geq 0$. Assume that there exist

1) a nonnegative function $V(s)$, $s \in S$;

2) a finite subset $S_0 \subset S$;

3) positive constants $T > 0$ and $\epsilon > 0$ such that

$$1^\circ.\ \inf_{s \in S \setminus S_0} V(s) > 0;$$
$$2^\circ.\ \sum_{r \in S} V(r) \mathrm{P}_{sr}^{V(s)T} \leqslant V(s)(1 - \epsilon),\ \forall s \in S \setminus S_0,$$

where P_{sr}^t, $s, r \in S$, $t \geq 0$, is the transition probability from state s to state r in time $t \geq 0$.

Then the Markov chain $s(t)$ is ergodic.

Let us consider in more detail the process that describes the behavior of this network. Let $Q(t) = \{Q_{ik}(t), (i, k) \in G, t \geq 0\}$, where $G \triangleq \{(i, k)\colon i \in I, k = 1, 2\}$, $Q_{ik}(t)$ is the total number of (i, k)-stream calls in node $j = \hat{j}(i, k)$ at time $t \geq 0$. Clearly $Q(t)$ is the projection of $s(t)$ (i.e., a function of the state $s(t)$). Define the norm of the state $s(t)$ as

$$\| s(t) \| \triangleq \| Q(t) \| \triangleq \sum_{(i, k) \in G} Q_{ik}(t).$$

All quantities relating to the process $s(t)$ will be equipped with the superscript n if

$$\|s(0)\| = n \geq 1.$$

Denote by $F_{ik}^{n}(t)$, $t \geq 0$, the total number of (i, k)-stream calls that arrive in node $j = \hat{j}(i, k)$ up to time t, including the (i, k)-stream calls that are in node j initially at time $t = 0$. For definiteness, let $F_{ik}^{n}(t)$ be left-continuous. Clearly, $F_{ik}^{n}(t)$ are nonnegative piecewise-constant nondecreasing functions. Define the negative constant T_0:

$$T_0 = -\min\,[\lambda_i^{-1},\, i \in I,\ v_{ik},\ (i, k) \in G].$$

Continue the functions $F_{ik}^{n}(t)$ to the interval $[nT_0, 0]$ in the following way. The calls initially present in the network at time $t = 0$ are assumed to have arrived at the negative times nT_0, $(n - 1)T_0$, ..., T_0 (one call at a time). The arrival sequence of the calls of different types corresponds to their order in the queues at time 0.

Denote by $\hat{F}_{ik}^{n}(t)$ the total number of (i, k)-stream calls that have left the node $j = \hat{j}(i, k)$ up to time t. For every $n \geq 1$ we obviously have

$$F_{ik}^{n}(n T_0) = 0,\ \hat{F}_{ik}^{n}(0) = 0,\ (i, k) \in G,$$
$$\sum_{(i, k) \in G} F_{ik}^{n}(0) = \|Q^{n}(0)\| \equiv n.$$

If the norm of the initial state of the network is $n \geq 1$, then consider the process

$$F^{n;(t)} \triangleq \{F_{ik}^{n}(\xi),\ \xi \in [nT_0, t],\ \hat{F}_{ik}^{n}(\xi'),\ \xi' \in [0, t],\ (i, k) \in G\},\ t \geqslant 0.$$

Remark. The following identities obviously hold for any $t \geq 0$:

$$F_{ik}^{n}(t) \equiv F_{ik}^{n}(0) + \hat{F}_{i,k-1}^{n}(t),\ i \in I,\ k = 2, \ldots, K(i).$$

The set of functions $F^{n;(t)}$ thus contains redundant information, but we use this definition from considerations of convenience.

The state of the process $F^{n;(t)}$ completely describes the behavior of the network up to time t. Thus, the state of the Markov process $s(t)$ is a function of $F^{n;(t)}$.

Define the following mapping of the family of processes $\{F^{n;(t)}\}$ with different n and different initial states $\{F^{n;(0)}\}$ to the family of normalized processes $\{f^{n;(t)}\}$. Let

$$f^{n;(t)} \triangleq \{f_{ik}^{n}(\xi),\ \xi \in [T_0, t],\ \hat{f}_{ik}^{n}(\xi'),\ \xi' \in [0, t],\ (i, k) \in G\},\ t \geqslant 0,$$

where

$$f_{ik}^{n}(t) \triangleq \frac{1}{n} F_{ik}^{n}(nt),\ t \geqslant T_0,$$

$$\hat{f}_{ik}^{n}(t) \triangleq \frac{1}{n} \hat{F}_{ik}^{n}(nt),\ t \geqslant 0.$$

Let

$$q_{ik}^{n}(t) \triangleq \frac{1}{n} Q_{ik}^{n}(nt),$$

$$q^{n}(t) \triangleq \{q_{ik}^{n}(t),\ (i, k) \in G\}.$$

Clearly, for every $n \geq 1$ and every $f^{n;(0)}$

$$f_{ik}^{n}(T_0) = 0,\ \forall\ (i, k) \in G,$$
$$\|q^{n}(0)\| = \sum_{(i, k) \in G} f_{ik}^{n}(0) = 1.$$

Take $\|s\|$ as the function $V(s)$ in the conditions of Theorem 2. To prove Theorem 1, it suffices to prove the following assertion.

There exist constants $T > 0$ and $\epsilon > 0$ such that for any sequence of initial states $F^{n;(0)}$, $n \geq 1$, of the processes $F^{n;(t)}$ for all $n \in \{1,2,\ldots\}$ with the possible exception of a finite subset we have the inequality

$$\mathbf{E}\,\| Q^n(nt) \| \leqslant \| Q^n(0) \| (1-\epsilon) = n(1-\epsilon).$$

Using the definition of the processes $f^{n;(t)}$, we can restate this property as follows.

There exist constants $T > 0$ and $\epsilon > 0$ such that for any sequence of initial states $f^{n;(0)}$, $n \geq 1$, of the processes $f^{n;(t)}$ we have the inequality

$$\lim_{n\to\infty} \sup \mathbf{E}\,\| q^n(T) \| \leqslant 1-\epsilon.$$

The following stronger result will be proved in Secs. 4 and 5.

THEOREM 3. There exists a constant $T > 0$ such that for any $t \geq T$ and any sequence of initial states $f^{n;(0)}$ of the processes $f^{n;(t)}$ we have the equality

$$\lim_{n\to\infty} \mathbf{E}\,\| q^n(t) \| = 0.$$

4. LIMIT DETERMINISTIC PROCESS AND ITS PROPERTIES

Take an arbitrary pair $(i, k) \in G$. Consider the sequence of functions $\{f_{ik}{}^n(t),\ t \in [T_0, 0]\}$ entering the corresponding initial states $f^{n;(0)}$, $n \geq 1$, in the conditions of Theorem 3. It is easy to see that this sequence has limit points in the sense of the uniform metric $\|g - g'\| \triangleq \sup_{t\in[T_0,0]} [g(t) - g'(t)]$, and every limit point $f_{ik}(t)$, $t \in [T_0, 0]$, is a nondecreasing continuous function that satisfies the Lipschitz condition with the constant $L \equiv |T_0|^{-1}$ and the condition

$$f_{ik}(T_0) = 0.$$

The sequence of initial states $f^{n;(0)}$, $n \geq 1$, thus contains a subsequence $f^{n_l;(0)}$, $l \geq 1$, that converges to the set of functions $f^0 = \{f_{ik}(t),\ t \in [T_0, 0],\ \hat{f}_{ik}(0) \equiv 0,\ (i, k) \in G\}$. Each of the functions $f_{ik}(t)$, $t \in [T_0, 0]$, $(i, k) \in G$, satisfies the above conditions, and

$$\sum_{(i,k)\in G} f_{ik}(0) = 1.$$

We naturally assume that for any fixed $t \geq 0$ the sequence of collections of random functions $f^{n_l;(t)}$ (or equivalently the sequence of stochastic processes $f^{n_l;(\xi)}$, $\xi \in [0, t]$) converges in probability to some set of deterministic functions $f^{(t)}$ (or to the deterministic process $f^{(\xi)}$, $\xi \in [0, t]$).

In this paper, we do not prove convergence of the processes. Instead, we provide a formal definition of the "limit" deterministic process $f^{(t)}$ that corresponds to the "limit" initial state $f^{(0)}$ and examine its properties. In the next section, these properties of the deterministic process are generalized in a certain sense to sequences of stochastic processes $f^{n;(\cdot)}$, $n \to \infty$. This generalization enables us to prove Theorem 3.

Consider the set of functions

$$f = \{ f_{ik}(t),\ t \geqslant T_0,\ \hat{f}_{ik}(t'),\ t' \geqslant 0,\ (i, k) \in G\},$$

which satisfies the following conditions.

1. For any $(i, k) \in G$, the functions $f_{ik}(t)$ and $\hat{f}_{ik}(t)$ are nondecreasing, continuous, and satisfy the Lipschitz condition with the constant $L = |T_0|^{-1}$, and $f_{ik}(T_0) = 0,\ \hat{f}_{ik}(0) = 0.$

2. $\sum_{(i,k)\in G} f_{ik}(0) = 1$.

Denote

$$w_j(t) \triangleq \sum_{(i,k)\in G_j} f_{ik}(t)\, v_{ik},\quad t \geqslant 0,\ j \in J,$$

$$\tilde{w}_j(t) \triangleq w_j(t) - t,$$

$$\hat{w}_j(t) \triangleq t + \min\,[0, \min_{0<\xi<t} [\tilde{w}(\xi)]],\quad t \geqslant 0, j \in J,$$

$$\tau_j(t) \triangleq \max\,[\tau \leqslant t\colon w_j(\tau) = \hat{w}_j(t)],\quad t \geqslant 0, j \in J.$$

We assume that the set of functions f satisfies also the following condition:

$$3.\ \hat{f}_{ik}(t)=f_{ik}(\tau_{\hat{j}(i,k)}(t)),\ t\geqslant 0,\ (i,k)\in G,$$
$$f_{ik}(t)=f_{ik}(0)+\lambda_i t,\ \ k=1,\ t\geqslant 0,\ i\in I,$$
$$f_{ik}(t)=f_{ik}(0)+\hat{f}_{i,k-1}(t),\ t\geqslant 0,\ k>1,\ i\in I.$$

Definition. If the set of functions f satisfies conditions 1-3, then

$$f^{(t)}=\{f_{ik}(\xi),\ \xi\in[T_0,t],\ \hat{f}_{ik}(\xi'),\ \xi'\in[0,t],\ \ (i,k)\in G\},\ \ t\geqslant 0,$$

is called the deterministic process corresponding to the initial state

$$f^{(0)}=\{f_{ik}(\xi),\ \ \xi\in[T_0,0],\ \hat{f}_{ik}(0)=0,\ (i,k)\in G\}.$$

THEOREM 4. For any initial state $f^{(0)}$ (i.e., a set of functions $\{f_{ik}(t),\ t\in[T_0,0],\ (i,k)\in G\}$ that satisfy conditions 1 and 2) there exists at least one corresponding deterministic process $f^{(t)}$, $t\geq 0$.

The proof easily follows, for instance, from Schauder's fixed-point theorem for a completely continuous operator (see [9]). We omit the details. If some deterministic process $f^{(t)}$, $t\geq 0$, has been fixed, then we naturally define

$$q(t)\triangleq\{q_{ik}(t),\ \ (i,k)\in G\},$$

where

$$q_{ik}(t)=f_{ik}(t)-\hat{f}_{ik}(t),$$
$$\|q(t)\|\triangleq\sum_{(i,k)\in G}q_{ik}(t).$$

In this section, we prove the following theorem for the deterministic process $f^{(t)}$, which is an analog of Theorem 3.

THEOREM 5. There exists a constant $T>0$ such that any deterministic process $f^{(t)}$ corresponding to an arbitrary initial state $f^{(0)}$ has the property

$$\|q(t)\|=0,\ \ \forall\, t\geqslant T.$$

To prove Theorem 5, we need two intuitively obvious lemmas. We introduce the additional definitions

$$q_j(t)\triangleq\{q_{ik}(t),\ (i,k)\in G_j\},$$
$$\|q_j(t)\|\triangleq\sum_{(i,k)\in G_j}q_{ik}(t),$$
$$\lambda_{ik}(t)\triangleq f'_{ik}(t),\ \ \hat{\lambda}_{ik}(t)\triangleq\hat{f}'_{ik}(t),\ t\geqslant 0,\ (i,k)\in G.$$

These derivatives exist almost everywhere (in the Lebesgue measure) on the interval $[0,\infty)$.

We agree that the notation

$$\lambda_{ik}(t)\geqslant a,\ \ \forall\, t\in[T_1,T_2],$$

indicates that the inequality holds almost everywhere on the interval $[T_1, T_2]$. Clearly

$$\lambda_{i1}(t)\equiv\lambda_i,\ \ \forall\, t\geqslant 0,\ i\in I.$$

Lemmas 1 and 2 consider an arbitrary fixed node $j\in J$. Denote for brevity $G_j=\{\alpha,\beta\}\subseteq I\times\{1,2\}$, i.e., two streams, an α-stream and a β-stream, arrive in node j.

LEMMA 1. For any constant $T_1\geq 0$ there exists a constant $T_2=T_2(T_1)$ such that for any constants ${}_*\lambda_\alpha$ and ${}^*\lambda_\beta$ and any deterministic process $f^{(t)}$, $t\geq 0$, the property

$$\begin{cases}\lambda_\alpha(t)\geqslant {}_*\lambda_\alpha,\\ \lambda_\beta(t)\leqslant {}^*\lambda_\beta,\end{cases}\qquad \forall\, t\geqslant T_1$$

implies the properties

$$\hat{\lambda}_\alpha(t)\geqslant\min\left[{}_*\lambda_\alpha,\ \frac{{}_*\lambda_\alpha}{{}_*\lambda_\alpha v_\alpha+{}^*\lambda_\beta v_\beta}\right],\ \forall\, t\geqslant T_2,\tag{2}$$

$$\hat{\lambda}_\beta \leq \frac{{}^{*}\lambda_\beta}{{}_{*}\lambda_\alpha v_\alpha + {}^{*}\lambda_\beta v_\beta}, \quad \forall t \geq T_2. \tag{3}$$

LEMMA 2. For any constant $T_1 \geq 0$ and any constants ${}^{*}\lambda_\alpha$ and ${}^{*}\lambda_\beta$ such that

$${}^{*}\rho_j \triangleq {}^{*}\lambda_\alpha v_\alpha + {}^{*}\lambda_\beta v_\beta < 1,$$

there exists a constant $T_3 = T_3(T_1, {}^{*}\rho_j)$ such that for any deterministic process $f^{(t)}$, $t \geq 0$, the property

$$\begin{cases} \lambda_\alpha(t) \leq {}^{*}\lambda_\alpha, \\ \lambda_\beta(t) \leq {}^{*}\lambda_\beta, \end{cases} \quad \forall t \geq T_1,$$

implies the property

$$\| q_j(t) \| = 0, \quad \forall t \geq T_3,$$

and thus

$$\begin{cases} \hat{\lambda}_\alpha(t) = \lambda_\alpha(t), \\ \hat{\lambda}_\beta(t) = \lambda_\beta(t), \end{cases} \quad \forall t \geq T_3.$$

Proof of Theorem 5. Write conditions (1) in explicit form:

$$\lambda_1 v_{11} + \lambda_2 v_{22} < 1, \tag{4}$$
$$\lambda_1 v_{12} + \lambda_2 v_{21} < 1. \tag{4'}$$

Consider an arbitrary deterministic process $f^{(t)}$, $t \geq 0$, corresponding to an arbitrary initial state $f^{(0)}$. Apply Lemmas 1 and 2. As we have noted above, $\lambda_{11}(t) = \lambda_1 = \text{const}$ and $\lambda_{21}(t) = \lambda_2 = \text{const}$ for all $t \geq 0$. Consider the following sequences of upper and lower bounds on the rates $\lambda_{12}(t)$ and $\lambda_{22}(t)$. Take

$${}_{*}\lambda_{12}^{(1)} = {}_{*}\lambda_{22}^{(1)} = 0, \quad {}^{*}\lambda_{12}^{(1)} = v_{11}^{-1}, \quad {}^{*}\lambda_{22}^{(1)} = v_{21}^{-1}.$$

Define recursively

$${}_{*}\lambda_{12}^{(m)} = \min\,[\lambda_1, \varphi_{12}({}^{*}\lambda_{22}^{(m-1)})], \tag{5}$$
$${}^{*}\lambda_{22}^{(m)} = \varphi_{22}({}_{*}\lambda_{12}^{(m-1)}), \tag{6}$$
$${}_{*}\lambda_{22}^{(m)} = \min\,[\lambda_2, \varphi_{22}({}^{*}\lambda_{12}^{(m-1)})], \tag{7}$$
$${}^{*}\lambda_{12}^{(m)} = \varphi_{12}({}_{*}\lambda_{22}^{(m-1)}), \quad m = 2, 3, \ldots, \tag{8}$$

where

$$\varphi_{12}(x) \triangleq \frac{\lambda_1}{\lambda_1 v_{11} + x v_{22}},$$

$$\varphi_{22}(x) \triangleq \frac{\lambda_2}{\lambda_2 v_{21} + x v_{12}}.$$

By Lemma 1, there exists an ascending sequence

$$0 < T_1 < T_2 < \ldots < T_m < \ldots,$$

such that

$$\begin{cases} {}_{*}\lambda_{12}^{(m)} \leq \lambda_{12}(t) \leq {}^{*}\lambda_{12}^{(m)}, \\ {}^{*}\lambda_{22}^{(m)} \geq \lambda_{22}(t) \geq {}_{*}\lambda_{22}^{(m)}, \end{cases} \quad \forall t \geq T_m.$$

Clearly, as $m \to \infty$,

$$
\begin{aligned}
&{}_{\bullet}\lambda_{12}^{(m)} \uparrow {}_{\bullet}\lambda_{12} \leqslant \lambda_1, \\
&{}_{\bullet}\lambda_{22}^{(m)} \uparrow {}_{\bullet}\lambda_{22} \leqslant \lambda_2, \\
&{}^{\bullet}\lambda_{12}^{(m)} \downarrow {}^{\bullet}\lambda_{12} \geqslant \lambda_1, \\
&{}^{\bullet}\lambda_{22}^{(m)} \downarrow {}^{\bullet}\lambda_{22} \geqslant \lambda_2.
\end{aligned}
$$

where the limits $\{{}_{*}\lambda_{12}, {}_{*}\lambda_{22}, {}^{*}\lambda_{12}, {}^{*}\lambda_{22}\}$ satisfy Eqs. (5)-(8) for $m = \infty$ and $m - 1 = \infty$. We have the following proposition.

LEMMA 3. At least one of the following inequalities is true:

$$\lambda_1 v_{11} + {}^{\bullet}\lambda_{22} v_{22} < 1, \tag{9}$$

$$\lambda_2 v_{21} + {}^{\bullet}\lambda_{12} v_{12} < 1. \tag{9'}$$

The proof is given in the Appendix. For instance, assume that inequality (9) is true. Then there exists m such that

$$\lambda_{11}(t) v_{11} + \lambda_{22}(t) v_{22} < \lambda_1 v_{11} + {}^{\bullet}\lambda_{22}^{(m)} v_{22} < 1, \ \forall t \geqslant T_m.$$

Then, by Lemma 2, there exists a constant $\tilde{T}$ such that

$$\| q_1(t) \| = 0, \lambda_{12}(t) = \lambda_{11}(t) = \lambda_1, \ \forall t \geqslant \tilde{T}.$$

Finally, using Lemma 2 for the node $j = 2$, we obtain that there exists a constant $T \geq \tilde{T}$ such that

$$\| q_2(t) \| = 0, \ \forall t \geqslant T.$$

Q.E.D.

5. PROOF OF THEOREM 3

The main idea of the proof of Theorem 3 repeats the outline of the proof of Theorem 5 while at the same time replacing the properties of the deterministic process $f^{(t)}$, $t \geq 0$, with the corresponding asymptotic properties of the sequence of stochastic processes $f^{n;(t)}$, $t \geq 0$, as $n \to \infty$.

We introduce the following notation. Assume that for an arbitrary function $A(t)$

$$A(t_1, t_2) \triangleq A(t_2) - A(t_1).$$

We agree that for $0 \leq T_1 \leq T_2 \leq \infty$ the expression

$$\lambda_{ik}^{\infty}(t) \geqslant \lambda, \ \forall t \in [T_1, T_2]$$

indicates that for any $t_1, t_2 \in [T_1, T_2]$, $t_1 \leq t_2$, and any $\epsilon > 0$

$$\lim_{n \to \infty} \mathrm{P}\{f_{ik}^{n}(t_1, t_2) \geqslant (\lambda - \epsilon)(t_2 - t_1)\} = 1.$$

Similarly the expression

$$\lambda_{ik}^{\infty}(t) \leqslant \lambda, \quad \forall t \in [T_1, T_2],$$

where $0 \leq T_1 \leq T_2 \leq \infty$ indicates that $\forall t_1, t_2 \in [T_1, T_2]$, $t_1 \leq t_2$, $\forall \epsilon > 0$,

$$\lim_{n \to \infty} \mathrm{P}\{ f_{ik}^{n}(t_1, t_2) \leqslant (\lambda + \epsilon)(t_2 - t_1) \} = 1.$$

The equality

$$\lambda_{ik}^{\infty}(t) = \lambda, \ \forall t \in [T_1, T_2], \ 0 \leqslant T_1 \leqslant T_2 \leqslant \infty,$$

correspondingly indicates that the inequalities

$$\lambda_{ik}^{\infty}(t) \leqslant \lambda, \quad \lambda_{ik}^{\infty}(t) \geqslant \lambda, \ \forall t \in [T_1, T_2]$$

are both true at the same time.

The corresponding inequalities and equalities for $\hat{\lambda}_{ik}^{\infty}(t)$ have the same meaning. We should only replace $f_{ik}^{n}(t_1, t_2)$ with $\hat{f}_{ik}^{n}(t_1, t_2)$ in all definitions. We will agree that the equality

$$\hat{\lambda}_{ik}^{\infty}(t) = \lambda_{ik}^{\infty}(t), \ \forall t \in [T_1, T_2], \ 0 \leqslant T_1 \leqslant T_2 \leqslant \infty,$$

indicates that

$$\forall t_1, t_2 \in [T_1, T_2], t_1 \leqslant t_2,$$
$$| f_{ik}^{n}(t_1, t_2) - \hat{f}_{ik}^{n}(t_1, t_2)| \xrightarrow{\mathbf{P}} 0, \ n \to \infty.$$

We have the following probabilistic analogs of Lemmas 1 and 2. Here we also consider a fixed node $j \in J$, $G_j = \{\alpha, \beta\}$, and the sequence of stochastic processes $f^{n;(t)}$, $t \geq 0$, from Theorem 3.

LEMMA 1′. For any constant $T_1 \geq 0$ there exists a constant $T_2 = T_2(T_1)$ such that if

$$\begin{cases} \lambda_{\alpha}^{\infty}(t) \geqslant {}_{*}\lambda_{\alpha}, \\ \lambda_{\beta}^{\infty}(t) \leqslant {}^{*}\lambda_{\beta}, \end{cases} \forall t \geqslant T_1,$$

for some constants ${}_{*}\lambda_{\alpha}$ and ${}^{*}\lambda_{\beta}$, then

$$\hat{\lambda}_{\alpha}^{\infty}(t) \geqslant \min \left[{}_{*}\lambda_{\alpha}, \frac{{}_{*}\lambda_{\alpha}}{{}_{*}\lambda_{\alpha} v_{\alpha} + {}^{*}\lambda_{\beta} v_{\beta}} \right], \ \forall t > T_2, \tag{2′}$$

$$\hat{\lambda}_{\beta}^{\infty}(t) \leqslant \frac{{}^{*}\lambda_{\beta}}{{}_{*}\lambda_{\alpha} v_{\alpha} + {}^{*}\lambda_{\beta} v_{\beta}}, \ \forall t > T_2. \tag{3′}$$

LEMMA 2′. For any constant $T_1 \geq 0$ and any constants ${}^{*}\lambda_{\alpha}$ and ${}^{*}\lambda_{\beta}$ such that

$${}^{*}\rho_j \triangleq {}^{*}\lambda_{\alpha} v_{\alpha} + {}^{*}\lambda_{\beta} v_{\beta} < 1,$$

there exists a constant $T_3 = T_3(T_1, {}^{*}\rho_j)$ such that the property

$$\begin{cases} \lambda_{\alpha}^{\infty}(t) \leqslant {}^{*}\lambda_{\alpha}, \\ \lambda_{\beta}^{\infty}(t) \leqslant {}^{*}\lambda_{\beta}, \end{cases} \forall t \geqslant T_1,$$

implies the property

$$\| q_j^n(t) \| \xrightarrow{\mathbf{P}} 0, n \to \infty, \ \forall t \geqslant T_3,$$

and thus,

$$\begin{cases} \hat{\lambda}_{\alpha}^{\infty}(t) = \lambda_{\alpha}^{\infty}(t) \leqslant {}^{*}\lambda_{\alpha}, \\ \hat{\lambda}_{\beta}^{\infty}(t) = \lambda_{\beta}^{\infty}(t) \leqslant {}^{*}\lambda_{\beta}, \end{cases} \forall t \geqslant T_3.$$

Replacing everywhere in the proof of Theorem 5 $\lambda_{ik}(t)$ and $\hat{\lambda}_{ik}(t)$ with $\lambda_{ik}^{\infty}(t)$ and $\hat{\lambda}_{ik}^{\infty}(t)$ respectively and using Lemmas 1′ and 2′ instead of Lemmas 1 and 2, we obtain that there exists a constant $T > 0$ such that

$$\| q^n(t) \| \xrightarrow{\mathbf{P}} 0, n \to \infty, \ \forall t \geqslant T.$$

The family of random variables $\| q^n(t) \|$, $n \geq 1$, is uniformly integrable for any $t \geq 0$ (see Lemma 9 in the Appendix). Thus

$$\lim_{n \to \infty} \mathbf{E} \| q^n(t) \| = 0, \ \forall t \geqslant T,$$

Q.E.D.

6. A NETWORK WITH A SPECIAL PRIORITY DISCIPLINE. COUNTEREXAMPLE FOR CONJECTURE 1

Consider the following service discipline at the network nodes. In node 1, type-2 calls (i.e., (2, 2)-stream calls) have absolute priority; conversely, in node 2, type-1 calls (i.e., (1, 2)-stream calls) have absolute priority.

It is easy to see that with this service discipline the behavior of the network is described by a simpler Markov chain $Q(t)$ with a countable state space: $Q(t) = \{Q_{ik}(t),\ (i, k) \in G\}$, $t \geq 0$. Note that the states of Q for which

$$\begin{cases} Q_{12} > 0, \\ Q_{22} > 0, \end{cases}$$

are inessential and they can be a priori excluded from the phase space, thus producing an irreducible Markov process $Q(t)$.

We will show that the Markov process $Q(t)$ is nonrecurrent for a wide range of parameters λ_i, $i = 1, 2$, and ν_{ik}, $(i, k) \in G$, that satisfy the conditions

$$\rho_j < 1, j = 1, 2.$$

We leave all previous definitions unchanged, except when specially qualified. We again start the analysis with a "limit" deterministic process.

The deterministic process $q(t) = \{q_{ik}(t),\ (i, k) \in G,\ t \geq 0\}$ that corresponds to the initial state $q(0)$, $\|q(0)\| = 1$, is defined (similarly to Sec. 4) as the projection of the deterministic process $f^{(t)}$, $t \geq 0$,

$$q_{ik}(t) = f_{ik}(t) - \hat{f}_{ik}(t), (i, k) \in G, t \geqslant 0,$$

with an arbitrary initial state $f^{(0)}$ satisfying the condition

$$f_{ik}(0) = q_{ik}(0), (i, k) \in G.$$

The deterministic process $f^{(t)}$, $t \geq 0$, is defined as in Sec. 4 (for the FCFS discipline) replacing condition 3 with the following condition 3_*:

$$\begin{aligned} &3_*.\ \hat{f}_{ik}(t) = f_{ik}(\tau_{ik}(t)), t \geqslant 0, (i, k) \in G, \\ &f_{i1}(t) = f_{i1}(0) + \lambda_i t, t \geqslant 0, i \in I, \\ &f_{ik}(t) = f_{ik}(0) + \hat{f}_{ik-1}(t), i \in I, k = 2, \end{aligned}$$

where

$$\begin{aligned} &\tau_{ik}(t) \triangleq \max\,[\tau \leqslant t: w_{ik}(t) = \hat{w}_{ik}(t)],\ (i, k) \in G, \\ &w_{ik}(t) \triangleq f_{ik}(t)\, v_{ik},\ (i, k) \in G, \\ &\hat{w}_{ik}(t) \triangleq t + \min\,[0, \min_{0 \leqslant \xi \leqslant t} \tilde{w}_{ik}(\xi)].\ (i, k) = (1,2), (2,2), \\ &\tilde{w}_{ik}(t) = w_{ik}(t) - t,\ (i, k) \in G, \\ &\hat{w}_{11}(t) = \hat{w}_1(t) - \hat{w}_{22}(t), \\ &\hat{w}_{21}(t) = \hat{w}_2(t) - \hat{w}_{12}(t) \end{aligned}$$

(the functions $\hat{w}_j(t)$, $j \in J$, are defined in Sec. 4).

Remark 1. In the case considered in this section, the process $q(t)$ also can be defined as the solution of the differential equation of the form

$$\frac{d}{dt} q(t) = A(q(t)),\ t \geqslant 0$$

with a discontinuous function $A(\cdot)$.

Remark 2. It is easy to see that there exist initial states $q(0)$ such that the corresponding process $q(t)$ is not unique. For instance, at least two different processes $q(t)$ correspond to an initial state such that

$$q_{11}(0) > 0, q_{21}(0) > 0, q_{12}(0) = q_{22}(0) = 0.$$

Consider our network with the following parameters:

$$\lambda_1 = \lambda_2 = 1,\ v_{12} = v_{22} = v_2 > 1/2,\ v_{11} = v_{21} = v_1,$$

where condition (1) is satisfied in the form $\nu_1 + \nu_2 < 1$. We thus have the inequality $0 \leq \nu_1 < 1 - \nu_2$.

THEOREM 6. Let the following initial state of the process $q(t)$ be fixed:

$$q_{11}(0) = 1,\ q_{12}(0) = q_{21}(0) = q_{22}(0) = 0.$$

Let $T = \nu_2/(1 - \nu_2) > 1$, $T_1 = \nu_1/(1 - \nu_1)$. Then the deterministic process $q(t)$ is uniquely defined on the interval $[0, T]$ and has the form

$$q_{11}(t) = \begin{cases} 1 - ((\upsilon_1)^{-1} - 1)\,t, & 0 \leqslant t \leqslant T_1, \\ 0, & T_1 \leqslant t \leqslant T, \end{cases}$$

$$q_{12}(t) = \begin{cases} (\upsilon_1^{-1} - \upsilon_2^{-1})\,t, \ 0 \leqslant t \leqslant T_1, \\ (\upsilon_2 - \upsilon_1)(1 - \upsilon_1)^{-1}\upsilon_2^{-1} - (\upsilon_2^{-1} - 1)\,t, \ T_1 \leqslant t \leqslant T, \end{cases}$$

$$q_{21}(t) = t,\ 0 \leqslant t \leqslant T,$$

$$q_{22}(t) \equiv 0,\ 0 \leqslant t \leqslant T.$$

Thus, at time T,

$$q_{21}(T) = T > 1,\ q_{11}(T) = q_{12}(T) = q_{22}(T) = 0.$$

The proof follows directly from the definition of the process $q(t)$.

We see that the state of the process $q(T)$ at time T is similar (apart from symmetry) to the state at time $t = 0$ and $\|q(T)\| = T > \|q(0)\| = 1$. Therefore, the process $q(t)$ with the initial state considered in Theorem 6 has the properties

$$\|q(t)\| \neq 0,\ t \geqslant 0,$$
$$\|q(t)\| \to \infty,\ t \to \infty.$$

We now return to consider the Markov process $Q(t) = \{Q_{ik}(t),\ (i, k) \in Q\}$. Take an integer $n > 0$ and a constant c, $1 < c < \nu_2(1 - \nu_2)^{-1}$. Choose an arbitrary initial state $Q(0)$ of the process $Q(t)$ such that $Q_{11}(0) \geq n$. Consider the stopping time

$$\tau = \min\ [t \geqslant 0, Q_{21}(t) \geqslant cn].$$

It is easy to see that always $\mathbf{P}\{\tau < \infty\} = 1$.

LEMMA 4. There exist constants $a > 0$ and $b > 0$ such that

$$\mathbf{P}(B) > 1 - ae^{-bn},$$

where $B \triangleq \{\|Q(t)\| > 0,\ 0 \leq t \leq \tau\}$.

The proof is given in the Appendix.

THEOREM 7. If the network parameters are such that

$$\lambda_1 = \lambda_2 = 1, \upsilon_{12} = \upsilon_{22} = \upsilon_2 > 1/2,\ \upsilon_{11} = \upsilon_{12} = \upsilon_1 > 0, \upsilon_1 + \upsilon_2 < 1,$$

then the Markov process $Q(t)$ is nonrecurrent.

Proof. Fix an arbitrary initial state $Q(0)$ of the process $Q(t)$ such that $Q_{11}(0) = n$. Consider the following nondecreasing sequence of stopping times:

$$\tau_l,\ l \geqslant 0:\ \tau_0 = 0,$$
$$\tau_{2k+1} = \min\ [t \geqslant \tau_{2k}:\ Q_{21}(t) \geqslant nc^{2k}],\ k \geqslant 0,$$
$$\tau_{2k+2} = \min\ [t \geqslant \tau_{2k+1}:\ Q_{11}(t) \geqslant nc^{2k+1}], k \geqslant 0.$$

Note that $\mathbf{P}\{\tau_k \to \infty, k \to \infty\} = 1$.

Consider the following sequence of events B_l:

$$B_l \triangleq \{\|Q(t)\| > 0, \tau_{l-1} \leqslant t \leqslant \tau_l\},\ l \geqslant 1.$$

Clearly,

$$\mathbf{P}\Big(\bigcap_{l=1}^{\infty} B_l\Big) \geqslant 1 - \sum_{l=1}^{\infty} (1 - \mathbf{P}(B_l)). \tag{10}$$

By Lemma 4, for sufficiently large n the right-hand side of (10) is greater than zero. But the event $\cap_{l=1}^{\infty} B_l$ is contained in the event $\{\|Q(t)\| > 0, t \geq 0\}$, and therefore for the initial state $Q(0)$ with a sufficiently large $Q_{11}(0) = n$,

$$\mathbf{P}\{\|Q(t)\| > 0, t \geq 0\} > 0.$$

Q.E.D.

We would like to acknowledge the useful comments of R. L. Dobrushin, Yu. M. Baryshnikov, A. A. Pukhal'skii, E. A. Pecherskii, and S. G. Foss.

APPENDIX

Proof of Theorem 2. Denote by $a(t)$, $t \geq 0$, the relevant continuous-time Markov chain. Note that condition 2° of the theorem implies that

$$\sum_{r \in S} \mathbf{P}^{\tau}_{sr} v(\cdot) < \infty, \ \forall \tau \geq 0, \ \forall s \in S_0. \tag{A.1}$$

Consider the embedded Markov chain $\hat{a}(n) \triangleq a(t_n)$, $n = 0,1,2,\ldots$, where

$$t_0 = 0,$$
$$t_{n+1} = t_n + \tau(\hat{a}(n)), n \geq 0,$$
$$\tau(s) \triangleq \begin{cases} \tau = \text{const} > 0, & s \in S_0, \\ v(s)\, T, & s \in S \setminus S_0. \end{cases}$$

Denote by $\hat{\mathbf{P}}_{sr}$, $s, r \in S$, the transition probabilities of the chain $\hat{a}(n)$. Then from conditions 1°, 2° and (A.1), we obtain the following properties of the chain $\hat{a}(n)$:

$$\inf_{s \in S \setminus S_0} [v(s)] > 0, \tag{A.2}$$

$$\sum_{r \in S} \hat{\mathbf{P}}_{sr} v(r) \leq v(s)(1 - \epsilon), \ \forall s \in S \setminus S_0, \tag{A.3}$$

$$\sum_{r \in S} \hat{\mathbf{P}}_{sr} v(r) < \infty, \ \forall s \in S_0. \tag{A.4}$$

From conditions (A.2)-(A.4) it follows that the Markov chain $\hat{a}(n)$ is ergodic (by the Moustafa–Foster ergodicity test [10]). Moreover, Coffman and Stolyar (E. G. Coffman, Jr. and A. L. Stolyar, "Polling server on a line, general independent service times," unpublished) have shown that (given conditions (A.2)-(A.4)) the stationary distribution $\hat{\mathbf{P}}_s$, $s \in S$, of the chain $\hat{a}(n)$ satisfies the condition

$$\sum_{s \in S} \hat{\mathbf{P}}_s v(s) < \infty.$$

From the last condition it follows that

$$\sum_{s \in S} \hat{\mathbf{P}}_s \tau(s) < \infty. \tag{A.5}$$

Take an arbitrary initial state $a(0) = s$ of the process $a(t)$. Consider an auxiliary semi-Markov process $\tilde{a}(t)$, $t \geq 0$, where $\tilde{a}(t) \triangleq a(t_n)$ for $t_n \leq t < t_{n+1}$. Then by the renewal theorem for semi-Markov processes (see [11])

$$\mathbf{E}(\tilde{R}(s)) = \Big(\sum_{r \in S} \hat{\mathbf{P}}_r \tau(r)\Big)(\tau(s))^{-1},$$

where

$$\tilde{R}(s) \triangleq \inf\{t_n : n \geq 1, \ \tilde{a}(t_n) = s\}.$$

Thus,

$$\mathrm{E}(\dot{\tilde{R}}(s)) < \infty \tag{A.6}$$

by (A.5).

The initial process $a(t)$ can be viewed as a regenerative process with regeneration instants identified, for instance, with the arrivals in state s. Then the random regeneration time is equal in distribution to the random variable

$$R(s) = \inf\{t > 0: a(t) = s, \ (\exists t_* < t: a(t_*) \neq s)\}.$$

From condition (A.6) we obviously have

$$\mathrm{E}(R(s)) < \infty.$$

The process $a(t)$ is thus ergodic. Q.E.D.

Proof of Lemmas 1 and 2. We denote the increment of an arbitrary function $A(t)$ by

$$A(t_1, t_2) \triangleq A(t_2) - A(t_1).$$

Also

$$w_{ik}(t) \triangleq f_{ik}(t)\, v_{ik},$$
$$\hat{w}_{ik}(t) \triangleq \hat{f}_{ik}(t)\, v_{ik}.$$

We first prove the following simple lemma.

LEMMA 5. Given are an arbitrary constant $T_1 \geq 0$ and the node $j \in J$. Then there exists a constant $T_2 = T_2(T_1)$ such that

$$\gamma_j(T_1) \triangleq \min\,[t \geqslant T_1 : \| q_j(t) \| = 0] \in [T_1, T_2].$$

Proof. The following three conditions are obviously equivalent for any $t \geq 0$:

$$\| q_j(t) \| = 0 \Leftrightarrow \tau_j(t) = t \Leftrightarrow \tilde{w}_j(t) = \min_{0 \leqslant \xi \leqslant t} \tilde{w}_j(\xi) \leqslant 0.$$

Also obviously

$$\min_{0 \leqslant \xi \leqslant T_1} \tilde{w}_j(\xi) \geqslant -T_1$$

and

$$\tilde{w}_j(t) \leqslant \| q(0) \| \max_{i \in I} \sum_{\{k : \hat{j}(i,k) = j\}} v_{ik} +$$
$$+ \Big(\sum_{(i,k) \in G_j} \lambda_i v_{ik} \Big) t - t.$$

Since $\| q(0) \| = 1$, $\Sigma_{(i,k) \in G_j} \lambda_i v_{ik} = \rho_j < 1$, the right-hand side of the last inequality tends to $-\infty$ as $t \to \infty$. It thus suffices to choose T_2 so that for $t = T_2$ the right-hand side of this inequality is less than $-T_1$. Q.E.D.

Proof of Lemma 1. Since the node j is fixed, we simplify the notation by omitting the subscript j of τ_j, w_j, $\hat{w}_j$ whenever there is no danger of confusion.

Choose $T_2 = T_2(T_1)$, where $T_2(\cdot)$ is the function from Lemma 5. Then $\tau(t) \geq T_1$ for all $t \geq T_2$.

Fix the interval $[t_1, t_2] \subset [T_2, \infty)$ and denote

$$\theta_1 = \min\{t \in [t_1, t_2] : \| q_j(t) \| = 0\},$$
$$\theta_2 = \max\{t \in [t_1, t_2] : \| q_j(t) \| = 0\},$$

and let $\theta_1 = \theta_2 = t_2$ if the set $\{\cdot\}$ is empty. Consider the most general case, when $t_1 < \theta_1 < \theta_2 < t_2$. (All degenerate cases are considered similarly.) Note that the intervals $[t_1, \theta_1]$ and $[\theta_2, t_2]$ are the busy intervals of the server at node j, i.e.,

$$\hat{w}(t_1, \theta_1) = w(\tau(t_1), \tau(\theta_1)) = \theta_1 - t_1,$$
$$\hat{w}(\theta_2, t_2) = w(\tau(\theta_2), \tau(t_2)) = t_2 - \theta_2.$$

For the interval $[\theta_1, \theta_2]$ we have

$$\hat{w}_\alpha(\theta_1, \theta_2) = w_\alpha(\theta_1, \theta_2),$$
$$\hat{w}_\beta(\theta_1, \theta_2) = w_\beta(\theta_1, \theta_2),$$

because $\tau(\theta_m) = \theta_m$, $m = 1, 2$.

Let us prove the bound (2). Indeed,

$$\frac{\hat{w}_\alpha(t_1, \theta_1)}{\theta_1 - t_1} = \frac{\hat{w}_\alpha(t_1, \theta_1)}{\hat{w}_\alpha(t_1, \theta_1) + \hat{w}_\beta(t_1, \theta_1)} =$$
$$= \frac{w_\alpha(\tau(t_1), \tau(\theta_1))}{w_\alpha(\tau(t_1), \tau(\theta_1)) + w_\beta(\tau(t_1), \tau(\theta_1))} \geqslant$$
$$\geqslant \frac{{}_*\lambda_\alpha v_\alpha \tau(t_1, \theta_1)}{{}_*\lambda_\alpha v_\alpha \tau(t_1, \theta_1) + {}^*\lambda_\beta v_\beta \tau(t_1, \theta_1)} = \frac{{}_*\lambda_\alpha v_\alpha}{{}_*\lambda_\alpha v_\alpha + {}^*\lambda_\beta v_\beta}.$$

Similarly for the interval $[\theta_2, t_2]$ we obtain

$$\frac{\hat{w}_\alpha(\theta_2, t_2)}{t_2 - \theta_2} \geqslant \frac{{}_*\lambda_\alpha v_\alpha}{{}_*\lambda_\alpha v_\alpha + {}^*\lambda_\beta v_\beta}.$$

For the interval $[\theta_1, \theta_2]$ we obtain

$$\frac{\hat{w}_\alpha(\theta_1, \theta_2)}{\theta_2 - \theta_1} = \frac{w_\alpha(\theta_1, \theta_2)}{\theta_2 - \theta_1} \geqslant {}_*\lambda_\alpha v_\alpha.$$

Thus,

$$\frac{\hat{w}_\alpha(t_1, t_2)}{t_2 - t_1} \geqslant \min\left\{{}_*\lambda_\alpha v_\alpha, \frac{{}_*\lambda_\alpha v_\alpha}{{}_*\lambda_\alpha v_\alpha + {}^*\lambda_\beta v_\beta}\right\},$$

whence follows (2), because the interval $[t_1, t_2]$ is arbitrary in $[T_2, \infty)$.

Let us prove the bound (3). For the intervals $[t_1, \theta_1]$ and $[\theta_2, t_2]$ we easily obtain the bound

$$\left.\begin{array}{l} \dfrac{\hat{w}_\beta(t_1, \theta_1)}{\theta_1 - t_1}, \\ \dfrac{\hat{w}_\beta(\theta_2, t_2)}{t_2 - \theta_2} \end{array}\right\} \leqslant \frac{{}^*\lambda_\beta v_\beta}{{}^*\lambda_\beta v_\beta + {}_*\lambda_\alpha v_\alpha} \equiv {}^*\hat{\lambda}_\beta v_\beta,$$

where ${}^*\hat{\lambda}_\beta$ stands for the right-hand side of inequality (3). Now obviously

$$\frac{\hat{w}_\beta(\theta_1, \theta_2)}{\theta_2 - \theta_1} = \frac{w_\beta(\theta_1, \theta_2)}{\theta_2 - \theta_1} \leqslant \min\{1 - {}_*\lambda_\alpha, v_\alpha, {}^*\lambda_\beta v_\beta\}.$$

We directly verify that

$${}^*\hat{\lambda}_\beta v_\beta \geqslant \min\{1 - {}_*\lambda_\alpha v_\alpha, {}^*\lambda_\beta v_\beta\}.$$

Thus,

$$\frac{\hat{w}_\beta(t_1, t_2)}{t_2 - t_1} \leqslant {}^*\hat{\lambda}_\beta v_\beta,$$

whence follows (3). Q.E.D.

Proof of Lemma 2. Take $T_3 = T_2(T_1)$, where $T_2(\cdot)$ is the function from Lemma 5. Thus, $\gamma_j(T_1) \le T_3$. But the function $\bar{w}_j(t)$ is strictly decreasing in the interval $[T_1, \infty)$. Thus, for all $t \ge \gamma_j(T_1)$

$$\tau_j(t) = t,$$

which is equivalent to the equality

$$\| q_j(t) \| = 0$$

and the condition

$$\begin{cases} \hat{f}_\alpha(t) = f_\alpha(t), \\ \hat{f}_\beta(t) = f_\beta(t). \end{cases}$$

Q.E.D.

Proof of Lemma 3. Without loss of generality, let $\lambda_1 = \lambda_2 = 1$. The proof of Lemma 3 follows from Lemmas 6-8.

LEMMA 6. Let $\nu_{11} = \nu_{21} = \nu_1$, $\nu_{12} = \nu_{22} = \nu_2$ (subject to $\nu_1 + \nu_2 < 1$). Then both inequalities (9) and (9′) hold.

Proof. By symmetry ${}_*\lambda_{12} = {}_*\lambda_{22}$ and ${}^*\lambda_{12} = {}^*\lambda_{22}$. Inequalities (9) and (9′) are thus simply identical.

Suppose that inequality (9′) does not hold, i.e., $\lambda_2\nu_{21} + {}^*\lambda_{12}\nu_{12} \ge 1$. Then (7) and (8) take the form

$${}_*\lambda_{22} = \varphi_{22}({}^*\lambda_{12}), \; {}^*\lambda_{12} = \varphi_{12}({}_*\lambda_{22}).$$

Using the conditions of the lemma, we obtain a quadratic equation for ${}_*\lambda_{22} \equiv z$:

$$z = \frac{1}{v_1 + v_2 (v_1 + v_2 z)^{-1}}.$$

Its unique positive root is

$$z = \frac{\sqrt{v_1^2 - 4v_2} - v_1}{2 v_2} > 1,$$

because

$$\frac{dz}{dv_1} = \frac{1}{2v_2} \left(\frac{v_1}{\sqrt{v_1^2 + 4v_2}} - 1 \right) < 0$$

for $0 \le \nu_1 \le 1 - \nu_2$ and $z = 1$ for $\nu_1 = 1 - \nu_2$.

Thus ${}_*\lambda_{22} > 1$, which contradicts Eq. (7). Q.E.D.

LEMMA 7. Consider the system with the parameters $\tilde{\nu}_{11} = \nu_1$, $\tilde{\nu}_{22} = \nu_2$, $\tilde{\nu}_{21} = \nu_1 + \alpha$, $\tilde{\nu}_{12} = \nu_2 - \alpha$, where $\nu_1 + \nu_2 < 1$, $0 \le \alpha < \min\{\nu_2, 1 - \nu_1\}$. Then both inequalities (9) and (9′) hold.

Proof. The system with the parameters $\nu_{11} = \nu_{21} = \nu_1$, $\nu_{12} = \nu_{22} = \nu_2$ will be called the original system. This system is considered in Lemma 6, and we retain the notation of Lemma 6. The system considered in this lemma is called a modified system and all the relevant variables and functions will be denoted by a tilde.

Note that $\varphi_{12}(x)$ and $\varphi_{22}(x)$ are decreasing functions. We also have the relationships

$$\tilde{\varphi}_{12}(x) \equiv \varphi_{12}(x),$$
$$\tilde{\varphi}_{22}(x) \geqslant \varphi_{22}(x), \; x \geqslant 1,$$

and

$$\tilde{\varphi}_{22}(x) = \frac{1}{(v_1 + \alpha) + (v_2 - \alpha)x} \leqslant \varphi_{22}(x), \; x \leqslant 1.$$

Since

$${}^*\tilde{\lambda}_{12}^{(1)} = {}^*\lambda_{12}^{(1)}, \; {}^*\tilde{\lambda}_{22}^{(1)} \leqslant {}^*\lambda_{22}^{(1)},$$

using the above relationships we obtain

$${}_*\tilde{\lambda}_{ik}^{(m)} \geqslant {}_*\lambda_{ik}^{(m)}, \; {}^*\tilde{\lambda}_{ik}^{(m)} \leqslant {}^*\lambda_{ik}^{(m)},$$

for all $m = 1,2,\ldots$, $(i, k) = (1, 2), (2, 2)$. Hence

$$^{*}\tilde{\lambda}_{ik} \leqslant {}^{*}\lambda_{ik}, (i, k) = (1, 2), (2, 2).$$

Since the conditions (9) and (9′) are satisfied for $^{*}\lambda_{12}$ and $^{*}\lambda_{22}$ (Lemma 6), they are also satisfied for $^{*}\tilde{\lambda}_{12}$ and $^{*}\tilde{\lambda}_{22}$. Q.E.D.

LEMMA 8. Assume that the condition (9) holds for the system with the parameters ν_{11}, ν_{12}, ν_{21}, ν_{22}. Then this condition also holds for the modified system with the parameters

$$\tilde{\upsilon}_{11} = \upsilon_{11} - \alpha, \tilde{\upsilon}_{22} - \alpha, \tilde{\upsilon}_{12} = \upsilon_{12}, \tilde{\upsilon}_{21} = \upsilon_{21},$$

where $0 \leq \alpha < \min\{\nu_{11}, \nu_{22}\}$.

Proof. Tilde denotes the quantities for the modified system. Clearly,

$$\tilde{\varphi}_{12}(x) \geqslant \varphi_{12}(x),$$
$$\tilde{\varphi}_{22}(x) \equiv \varphi_{22}(x), x \geqslant 0.$$

Then arguing as in the proof of Lemma 7, we obtain the inequality

$$^{*}\tilde{\lambda}_{22} \leqslant {}^{*}\lambda_{22}.$$

Inequality (9) thus holds for $^{*}\lambda_{22}$. Q.E.D.

Proof of Lemmas 1′ and 2′. We will require some additional notation.

For the original unnormalized process $F^{n;(t)}$, $t \geq 0$ (with a fixed norm of the initial state $\|Q^n(0)\| = n \geq 1$), we define the following quantities ($t \geq 0$ everywhere):

$W_{ik}{}^{n}(t)$ is the total service time for all (i, k)-stream calls arriving in the node $j = \hat{j}(i, k)$ up to time t (i.e., in the interval $[nT_0, t)$), $(i, k) \in G$;

$$W_j^n(t) \triangleq \sum_{(i, k) \in G_j} W_{ik}^n(t), \ j \in J;$$

$\hat{W}_{ik}{}^{n}(t)$ is the total time spent by the server in node $j = \hat{j}(i, k)$ serving (i, k)-stream calls up to time t (i.e., in the interval $[0, t)$), $(i, k) \in G$;

$$\hat{W}_j^n(t) \triangleq \sum_{(i, k) \in G_j} \hat{W}_{ik}^n(t), j \in J;$$

$T_j^n(t) \in [nT_0, t]$ is the arrival time in node j of the call which is being served at time t. If node j is empty at time t, then $T_j^n(t) = t$.

Note that the set of functions $W_{ik}{}^{n}(t)$, $t \geq nT_0$, $(i, k) \in G$, uniquely defines the sets of functions $\hat{W}_{ik}{}^{n}(t)$, $t \geq 0$, $(i, k) \in G$, and $T_j^n(t)$, $t \geq 0, j \in J$.

The corresponding quantities are similarly defined for the normalized process $f^{n;(t)}$, $t \geq 0$:

$$w_{ik}^n(t) \triangleq \frac{1}{n} W_{ik}^n(nt), \ w_j^n(t) \triangleq \sum_{(i, k) \in G_j} w_{ik}^n(t),$$

$$\hat{w}_{ik}^n(t) \triangleq \frac{1}{n} \hat{W}_{ik}^n(nt), \ \hat{w}_j^n(t) \triangleq \sum_{(i, k) \in G_j} \hat{w}_{ik}^n(t),$$

$$\tau_j^n(t) \triangleq \frac{1}{n} T_j^n(nt) \in [T_0, t].$$

In what follows we assume that the conditions of Theorem 3 are satisfied, i.e., we consider a sequence of stochastic processes $f^{n;(t)}$, $t \geq 0$, $n = 1,2,\ldots$, with fixed initial states $f^{n;(0)} = (f^n(t), t \in [T_0, 0])$.

We agree that the expression

$$\mu_{ik}^{\infty}(t) \geqslant \mu, \ \forall t \in [T_1, T_2],$$

where $0 \leq T_1 \leq T_2 \leq \infty$, stands for

$$\forall t_1, t_2 \in [T_1, T_2], t_1 \leqslant t_2, \ \forall \epsilon > 0 \cdot$$
$$\lim_{n \to \infty} \mathrm{P}\{\hat{w}_{ik}^n(t_1, t_2) \geqslant (\mu - \epsilon)(t_2 - t_1)\} = 1.$$

We similarly interpret the converse inequality

$$\mu_{ik}^{\infty}(t) \leq \mu, \ \forall\, t \in [T_1, T_2].$$

In the notation $P(C^n) \to 1$, where C^n, $n \geq 1$, is a sequence of events, it is always implied that $n \to \infty$.

We also use the notation

$$(C_1 \Rightarrow C_2) \triangleq \bar{C}_1 \cup (C_1 \cap C_2) \equiv \overline{(C_1 \cap \bar{C}_2)},$$

where C_1 and C_2 are events, $\bar{C}$ is the complement of the event C.

It is easy to verify the following propositions:

a) $P(C_1^n \Rightarrow C_2^n) \to 1$ and $P(C_2^n \Rightarrow C_3^n) \to 1$ implies that $P(C_1^n \Rightarrow C_3^n) \to 1$; and in particular,

b) $P(C_1^n) \to 1$ and $P(C_1^n \Rightarrow C_2^n) \to 1$ implies that $P(C_2^n) \to 1$;

c) $P(C_2^n) \to 1$ implies that for any sequence of events C_1^n, $n \geq 1$,

$$P(C_1^n \Rightarrow (C_1^n \cap C_2^n)) \to 1;$$

d) $P(C_1^n \Rightarrow C_2^n) \to 1$ and $C_2^n \subseteq C_3^n$, $n \geq 1$, implies that

$$P(C_1^n \Rightarrow C_3^n) \to 1.$$

LEMMA 9. For any $t \geq 0$ the family of random variables $\| q^n(t) \|$, $n \geq 1$, is uniformly integrable.

Proof. Clearly,

$$\| q^n(t) \| \leq \| q^n(0) \| + \sum_{i \in I} f_{i1}^n(0, t),$$

where $\| q^n(0) \| = 1$, $n \geq 1$, and $\Sigma_{i \in I} f_{i1}^n(0, t) = \Sigma_{i \in I} F_{i1}^n(0, nt)$ is the total number of calls that arrive in the system in the interval $[0, nt]$. This is a Poisson stream of arrivals with rate $\Sigma_{i \in I} \lambda_i$. Hence,

$$E\,[\Sigma f_{i1}^n(0, t)] = \Sigma \lambda_i t = \text{const},$$

$$D\,[\Sigma f_{i1}^n(0, t)] = \frac{1}{n} \Sigma \lambda_i t \to 0, \ n \to \infty,$$

which proves the lemma.

Lemma 5′ is a stochastic analog of Lemma 5.

LEMMA 5′. Let the constant $T_1 > 0$ and the node $j \in J$ be given. Then there exists a constant $T_2 = T_2(T_1)$ such that

$$P\{\gamma_j^n(T_1) \triangleq \min\{t \geq T_1 : \| q_j^n(t) \| = 0\} \in [T_1, T_2]\} \to 1.$$

Proof. The constant T_2 is chosen as in Lemma 5. We easily see that by the law of large numbers

$$P\{\tilde{w}_j^n(T_2) < -T_1 \leq \inf_{0 \leq \xi \leq T_1} \tilde{w}_j^n(\xi)\} \to 1,$$

where $\tilde{w}_j^n(t) \triangleq w_j^n(t) - t$, which proves the lemma.

We will need three technical lemmas, Lemmas 10-12. Their proof follows from the law of large numbers.

LEMMA 10. There exists a universal constant $\nu > 0$ such that for any fixed t_1 and t_2, $0 \leq t_1 \leq t_2$,

$$P\{\sum_{(i,k) \in G} (| f_{ik}^n(t_1, t_2)| + |\hat{f}_{ik}(t_1, t_2)|) < \nu(t_2 - t_1)\} \to 1,$$

$$P\{\sum_{(i,k) \in G} | w_{ik}^n(t_1, t_2)| < \nu(t_2 - t_1)\} \to 1,$$

$$P\{\sup_{t_1 \leq \xi_1 \leq \xi_2 \leq t_2} \| q^n(\xi_2) - q^n(\xi_1) \| < \nu(t_2 - t_1)\} \to 1.$$

In Lemmas 11 and 12, the node j is fixed and $G_j = \{\alpha, \beta\}$ (as in Lemmas 1′ and 2′).

LEMMA 11. Consider a fixed time interval $[t_1, t_2]$ and the constants r, $0 \leq r \leq t_2 - t_1$, and $\epsilon > 0$. Then

$$P\{(\hat{w}_j^n(t_1, t_2) \geq r) \Rightarrow (u_j^n(t_1, t_2) \geq r - \epsilon)\} \to 1. \tag{A.7}$$

$$P\{(u_j^n(t_1, t_2) \geq r) \Rightarrow (\hat{w}_j^n(t_1, t_2) \geq r - \epsilon)\} \to 1, \tag{A.8}$$

$$P\{(\hat{w}_j^n(t_1, t_2) \leq r) \Rightarrow (u_j^n(t_1, t_2) \leq r + \epsilon)\} \to 1, \tag{A.9}$$

$$\mathbf{P}\{(u_j^n(t_1,t_2)\leq r)\Rightarrow(\hat{w}_j^n(t_1,t_2)\leq r+\epsilon)\}\to 1, \tag{A.10}$$

where $u_j^n(t) \triangleq \sum_{\gamma=\alpha,\beta} \hat{f}_\gamma^n(t)\,v_\gamma$.

LEMMA 12. The following two assertions are equivalent ($\gamma = \alpha, \beta$):

$$\hat{\lambda}_\gamma^\infty(t)\geq\lambda,\ \forall t\in[T_1,T_2],$$

and

$$\mu_\gamma^\infty(t)\geq\lambda v_\gamma,\ \forall t\in[T_1,T_2].$$

Similarly, two assertions with the converse inequality are also equivalent.

Proof of Lemma 1'. Fix the constant $c > 0$ and choose the constant T_2 so that $T_2 = T_2'(T_1 + c) + c$, where $T_2'(\cdot)$ is the function $T_2(\cdot)$ from Lemma 5'. Thus, for any $t \geq T_2 - c$,

$$\mathbf{P}(A_1^n)\to 1,$$

where

$$A_1^n \triangleq \{\tau_j^n(t)\geq T_1+c\}.$$

Fix t_1 and t_2, $T_2 \leq t_1 \leq t_2$. Let $\epsilon_1 = (t_2 - t_1)/N$, where N is a positive integer. Consider the finite set of points $\Theta \triangleq \{\theta_l = t_1 + l\epsilon_1,\ l = 0,1,\ldots,N\}$, so that $\theta_0 = t_1$, $\theta_N = t_2$.

Let $\epsilon_2 = (\nu + 1)\epsilon_1$, where ν is the constant from Lemma 10. Let

$$\theta_* = \min\{\theta_l\in\Theta:\ \|q_j^n(\theta_l)\|<\epsilon_2\},$$
$$\theta^* = \max\{\theta_l\in\Theta:\ \|q_j^n(\theta_l)\|<\epsilon_2\},$$

and let $\theta_* = \theta^* = t_2$ if the set $\{\cdot\}$ is empty. By Lemma 10 and the choice of ϵ_2, for any $l = 0,1,\ldots,N$,

$$\mathbf{P}(A_{2,l}^n)\to 1,$$

where

$$A_{2,l}^n \triangleq \{(\|q_j^n(\theta_l)\|\geq\epsilon_2)\Rightarrow(\inf_{\theta_l-\epsilon_1\leq t\leq\theta_l+\epsilon_1}\|q_j^n(t)\|>0)\}.$$

Denote

$$z_j^n(t)\triangleq\frac{1}{n}Z_j^n(nt),\ t\geq 0,\ n\geq 1,$$

where $Z_j^n(t)$ is defined for the process $F^{n,(t)}$ as the total remaining service time in node j for all calls that reside at that node at time t. Fix an arbitrary constant $\nu_2 > \max\{v_\alpha, v_\beta\}$. Then by the law of large numbers we easily obtain

$$\mathbf{P}(A_{3,l}^n)\to 1,$$

where

$$A_{3,l}^n \triangleq \{(\|q_j^n(\theta_l)\|<\epsilon_2)\Rightarrow(z_j^n(\theta_l)<\epsilon_2\nu_2)\}.$$

Without loss of generality we assume that $\nu = \nu_2$.

Now fix the constant $\epsilon_3 > 0$ and consider the finite set $\Psi = \{\psi_l = l\epsilon_3,\ l = 0,1,\ldots,N_3 = [t_2/\epsilon_3] + 1\}$. From the conditions of the lemma it follows that for any constants ϵ_α, $\epsilon_\beta > 0$ and for any element $\psi_l \in \Psi$ such that $T_1 \leq \psi_l \leq (N_3 - 1)\epsilon_3$ we have

$$\mathbf{P}(A_{4,l}^n)\to 1,\ A_{4,l}^n\triangleq\{f_\alpha^n(\psi_l,\psi_{l+1})>({}_*\lambda_\alpha-\epsilon_\alpha)\epsilon_3\},$$
$$\mathbf{P}(A_{5,l}^n)\to 1,\ A_{5,l}^n\triangleq\{w_\alpha^n(\psi_l,\psi_{l+1})>({}_*\lambda_\alpha-\epsilon_\alpha)\epsilon_3 v_\alpha\},$$
$$\mathbf{P}(A_{6,l}^n)\to 1,\ A_{6,l}^n\triangleq\{f_\beta^n(\psi_l,\psi_{l+1})<({}^*\lambda_\beta+\epsilon_\beta)\epsilon_3\},$$
$$\mathbf{P}(A_{7,l}^n)\to 1,\ A_{7,l}^n\triangleq\{w_\beta^n(\psi_l,\psi_{l+1})<({}^*\lambda_\beta+\epsilon_\beta)\epsilon_3 v_\beta\}.$$

Moreover, for any ψ_l defined above, we have by Lemma 10

$$\mathbf{P}(A^n_{8,l}) \to 1, \; A^n_{8,l} \triangleq \{ \sum_{\gamma=\alpha,\beta} f^n_\gamma(\psi_l, \psi_{l+1}) < \nu\epsilon_3 \},$$

$$\mathbf{P}(A^n_{9,l}) \to 1, \; A^n_{9,l} \triangleq \{ w^n_j(\psi_l, \psi_{l+1}) < \nu\epsilon_3 \}.$$

Consider the event

$$B^n \triangleq \bigcap_{m,l} A^n_{m,l}.$$

In view of the above, $\mathbf{P}(B^n) \to 1$. Consider an arbitrary fixed realization of the process contained in the event B^n. For instance, assume that the most general case $t_1 < \theta_* < \theta^* < t_2$ applies.

Consider the busy interval $[t_1, \theta_*]$ (an analog of the interval $[t_1, \theta_1]$ in the proof of Lemma 1). Clearly

$$\hat{w}^n_\alpha(t_1, \theta_*) + \hat{w}^n_\beta(t_1, \theta_1) = \theta_* - t_1 \geqslant \epsilon_1.$$

Then for a sufficiently small ϵ_3,

$$\Delta\tau \triangleq \tau^n_j(\theta_*) - \tau^n_j(t_1) \geqslant \epsilon_1/\nu',$$

where $\nu' > \nu$ is a fixed constant. Indeed,

$$\epsilon_1 \leqslant \hat{w}^n_\alpha(t_1, \theta_*) + \hat{w}^n_\beta(t_1, \theta_*) \leqslant (\Delta\tau + 2\epsilon_3)\nu,$$

whence $\Delta\tau \geq \epsilon_1/\nu - 2\epsilon_3$. We thus obtain that for a given ϵ_1 and an arbitrary fixed $\epsilon_4 > 0$ we have the bound

$$\begin{aligned}\frac{\hat{w}^n_\alpha(t_1, \theta_*)}{\theta_* - t_1} &= \frac{\hat{w}^n_\alpha(t_1, \theta_*)}{\hat{w}^n_\alpha(t_1, \theta_*) + \hat{w}^n_\beta(t_1, \theta_*)} \geqslant \\ &\geqslant \frac{({}_*\lambda_\alpha - \epsilon_\alpha)\, v_\alpha (\Delta\tau - 2\epsilon_3)}{({}_*\lambda_\alpha - \epsilon_\alpha)\, v_\alpha (\Delta\tau - 2\epsilon_3) + ({}^*\lambda_\beta + \epsilon_\beta)\, v_\beta (\Delta\tau + 2\epsilon_3)} \geqslant \\ &\geqslant \frac{{}_*\lambda_\alpha v_\alpha}{{}_*\lambda_\alpha v_\alpha + {}^*\lambda_\beta v_\beta} - \epsilon_4,\end{aligned}$$

if ϵ_α, ϵ_β, and ϵ_3 are sufficiently small. A similar bound holds in the interval $[\theta^*, t_2]$.

Consider the interval $[\theta_*, \theta^*]$. Note that

$$|\hat{w}^n_\alpha(\theta_*, \theta^*) - w^n_\alpha(\theta_*, \theta^*)| < \epsilon_2 \nu.$$

We obtain that for the previously chosen ϵ_1 and an arbitrary fixed $\epsilon_5 > 0$ we have the bound ($\Delta\theta \triangleq \theta^* - \theta_*$):

$$\hat{w}^n_\alpha(\theta_*, \theta^*) \geqslant ({}_*\lambda_\alpha - \epsilon_\alpha)\, v_\alpha \cdot \Delta\theta - \epsilon_2\nu \geqslant {}_*\lambda_\alpha v_\alpha \Delta\theta - \epsilon_5 - \nu(\nu+1)\epsilon_1,$$

if ϵ_α is chosen sufficiently small.

We finally obtain the bound

$$\hat{w}^n_\alpha(t_1, t_2) \geqslant {}_*\hat{\lambda}_\alpha v_\alpha (t_2 - t_1) - \nu(\nu+1)\epsilon_1 - \epsilon_5 - \epsilon_4(t_2 - t_1),$$

where

$${}_*\hat{\lambda}_\alpha \triangleq \left\{ {}_*\lambda_\alpha, \; \frac{{}_*\lambda_\alpha}{{}_*\lambda_\alpha v_\alpha + {}^*\lambda_\beta v_\beta} \right\}.$$

The last bound indicates that

$$\mu^\infty_\alpha(t) \geqslant {}_*\hat{\lambda}_\alpha v_\alpha, \; \forall\, t \geqslant T_2,$$

because it holds for all realizations contained in the event B^n, $\mathbf{P}(B^n) \to 1$, and ϵ_1, ϵ_4, and ϵ_5 are arbitrary. By Lemma 12, this assertion is equivalent to the assertion

$$\hat{\lambda}^\infty_\alpha(t) \geqslant {}_*\hat{\lambda}_\alpha, \; \forall\, t \geqslant T_2.$$

This completes the proof of (2′). We similarly prove (3′). Q.E.D.

Proof of Lemma 2'. Fix the constant $c > 0$ and let $T_3 = T_2'(T_1) + c$, where $T_2'(\cdot)$ is the function $T_2(\cdot)$ constructed in Lemma 5'. Fix an arbitrary time $t \geq T_3$ and an arbitrary constant $\delta > 0$. Take a large natural number N such that

$$\epsilon \triangleq (t - T_1)/N < \delta/(2\nu),$$

where ν is the constant from Lemma 10. Consider the finite set $\Theta = \{\theta_l = T_1 + l\epsilon,\ l = 0,1,\ldots,N\}$, where obviously $\theta_0 = T_1$, $\theta_N = t$. Consider the following norm of the state of the node j:

$$\| q_j^n(\cdot) \|_* \triangleq \sum_{j=\alpha,\beta} q_j^n(\cdot) v_j .$$

Then for any $\theta_l \in \Theta$ we have

$$\mathrm{P}\{(\| q_j^n(\theta_l) \|_* \geqslant \epsilon\nu) \Rightarrow (\inf_{[\theta_l, \theta_{l+1}]} \| q_j^n(\xi) \|_* > 0)\} \to 1.$$

(The constant ν in Lemma 10 obviously can be chosen so that the third assertion is also true for the norm $\|\cdot\|_*$.) But

$$\{\inf_{[\theta_l, \theta_{l+1}]} \| q_j^n(\xi) \|_* > 0\} \subseteq \{\hat{w}_j^n(\theta_l, \theta_{l+1}) = \epsilon\},$$

and from Lemma 11 it follows that for any fixed $\epsilon_3 > 0$

$$\mathrm{P}\{(\hat{w}_j^n(\theta_l, \theta_{l+1}) = \epsilon) \Rightarrow (u_j^n(\theta_l, \theta_{l+1}) \geqslant \epsilon - \epsilon_3)\} \to 1.$$

Thus,

$$\mathrm{P}\{(\| q_j^n(\theta_l) \|_* \geqslant \epsilon\nu) \Rightarrow (\sum_{\gamma=\alpha,\beta} v_\gamma \hat{f}_\gamma^n(\theta_l, \theta_{l+1}) \geqslant \epsilon - \epsilon_3)\} \to 1.$$

Finally, we obtain that for any $\epsilon_\alpha, \epsilon_\beta > 0$:

$$\mathrm{P}(A_{1,l}^n) \to 1,$$

where

$$\begin{aligned} A_{1,l}^n \triangleq \{&(\| q_j^n(\theta_l) \|_* \geqslant \epsilon\nu) \Rightarrow \\ &\Rightarrow (\| q_j^n(\theta_{l+1}) \|_* \leqslant \| q_j^n(\theta_l) \|_* - (1 - {}^*\lambda_\alpha v_\alpha - {}^*\lambda_\beta v_\beta)\epsilon + \\ &+ \epsilon_3 + (\epsilon_\alpha v_\alpha + \epsilon_\beta v_\beta)\epsilon)\} \end{aligned}$$

Now

$$\mathrm{P}(A_{2,l}^n) \to 1,$$

where

$$A_{2,l}^n \triangleq \{(\| q_j^n(\theta_l) \|_* < \epsilon\nu) \Rightarrow (\sup_{[\theta_l, \theta_{l+1}]} \| q_j^n(\xi) \|_* < 2\epsilon\nu < \delta)\}.$$

Moreover, by the property (Lemma 5')

$$\mathrm{P}\{\gamma_j^n(T_1) \in [T_1, T_3 - c]\} \to 1$$

it follows that

$$\mathrm{P}(A_3^n) \to 1,$$

where

$$A_3^n \triangleq \{\exists\, \theta_l \in \Theta : \| q_j^n(\theta_l) \|_* < \epsilon\nu\}.$$

Consider the event

$$B^n \triangleq \bigcap_{m,l} A_{m,l}^n, \quad \mathrm{P}(B^n) \to 1.$$

Consider an arbitrary realization of the process contained in the event B^n. For this realization, the sequence of numbers $(\| q_j^n(\theta_l) \|_* \triangleq r_l,\ l = 0,1,\ldots,N)$ has the following properties:

a) $\exists l$: $r_l < \epsilon\nu$;

b) if $r_l < \epsilon\nu$, then $r_{l+1} < \delta$;

c) if $r_l \geq \epsilon\nu$, then $r_{l+1} < r_l$.

Hence $r_N \equiv \|q_j^n(t)\|_* < \delta$. We have thus obtained that

$$P\{\|q_j^n(t)\|_* < \delta\} \to 1.$$

This proves the first assertion of the lemma, which obviously leads to the second assertion.

Proof of Lemma 4. Consider the projection

$$\bar{Q}(t) = \{Q_{11}(t), Q_{12}(t)\}, \; t \geq 0,$$

of the process $Q(t)$. The process $\bar{Q}(t)$ describes the evolution of the number of type-1 calls in the network. Recalling that $Q_{12}(t)$ and $Q_{22}(t)$ do not both vanish at the same time, we can make the following comments. First, if $Q_{12}(t) > 0$ at some t_0, then the process $\bar{Q}(t)$ is Markov in the interval $[t_0, t_*]$, where $t_* \triangleq \min\{t \geq t_0: Q_{12}(t) = 0\}$, and it behaves in this interval as if there were no type-2 calls. Second, $Q_{11}(t)$ is nondecreasing on those time intervals where $Q_{22}(t) > 0$.

Consider the process $\tilde{Q}(t) = (\tilde{Q}_{11}(t), \tilde{Q}_{12}(t))$, $t \geq 0$, which is obtained from the process $\bar{Q}(t)$ by eliminating the time intervals where $Q_{22}(t) > 0$. Formally, let

$$\tilde{Q}(t) \triangleq \bar{Q}(\theta(t)),$$

where

$$\theta(t) = \sup\{\theta : \int_0^\theta I\{Q_{22}(\xi) = 0\} d\xi = t\}.$$

We obviously have the following properties:

a) $\theta(t) \geq t$;

b) if $\|\tilde{Q}(t)\| > 0$ for $t \in [0, \xi]$, then $\|\bar{Q}(t)\| > 0$ (and thus $\|Q(t)\| > 0$) for $t \in [0, \theta(\xi)]$;

c) if $\tilde{Q}_{12}(t) > 0$ for $t \in [\xi_1, \xi_2]$, then $Q_{12}(t) > 0$ for $t \in [\theta(\xi_1), \theta(\xi_2)]$, and $\theta(\xi_2) - \theta(\xi_1) = \xi_2 - \xi_1$.

The process $\tilde{Q}(t)$ may be viewed as the process that describes the evolution of our network as a result of the following modification:

1) there are no type-2 calls;

2) $\tilde{Q}_{11}$ may experience additional random jumps (i.e., independent of type-1 arrivals), and these jumps are positive and may occur only at time instants t when $\tilde{Q}_{12}(t) = 0$.

Finally, denote by $Q^n(t) = (Q_{11}{}^n(t), Q_{12}{}^n(t))$, $t \geq 0$, the Markov process with the initial state $Q_{11}{}^n(0) = n$, $Q_{12}{}^n(0) = 0$, which describes the evolution of our network without type-2 calls.

The processes $\tilde{Q}(t)$ and $Q^n(t)$ clearly can be constructed on one probability space so that

$$Q_{11}^n(t) \leq \tilde{Q}_{11}(t), \; Q_{12}^n(t) \leq \tilde{Q}_{12}(t),$$

for all $t \geq 0$ and all realizations of the processes.

Consider the sequence of processes

$$q^n(t) = (q_{11}^n(t), q_{12}^n(t)) \triangleq \frac{1}{n} Q^n(nt), \; 0 \leq t \leq T, \; n \geq 1.$$

Using upper bounds on the probabilities of large deviations for Markov processes describing the behavior of Jackson networks of queues [12], we obtain the following bound:

for any $\epsilon > 0$,

$$P(A_1^n) \leq a_1 e^{-b_1 n}, \; n \geq 1,$$

where

$$A_1^n \triangleq \{\sup_{0 \leq t \leq T} \|q^n(t) - q(t)\| \geq \epsilon\},$$

$$b_1 = b_1(\epsilon) > 0, \; q(t) = (q_{11}(t), q_{12}(t)),$$

and $q_{11}(\cdot)$ and $q_{12}(\cdot)$ are the deterministic functions from Theorem 6. For any $\epsilon_1 > 0$ and $\epsilon_2 > 0$, we can choose $\epsilon > 0$ such that

$$\{\exists\, t \in [0, T-\epsilon_2] : \| q^n(t) \| = 0\} \subseteq A_1^n,$$
$$\{\exists\, t \in [\epsilon_1, T-\epsilon_2] : q_{12}^n = 0\} \subseteq A_1^n,$$

where $T = v_2/(1 - v_2)$.
Thus,

$$\mathbf{P}\{\| Q^n(t) \| > 0,\ t \in [0, n(T-\epsilon_2)],$$
$$Q_{12}^n(t) > 0,\ t \in [n\epsilon_1, n(T-\epsilon_2)]\} \geqslant 1 - a_1 e^{-b_1 n},\ n \geqslant 1.$$

Using our preliminary arguments, we obtain

$$\mathbf{P}(A_2^n) \geqslant 1 - a_1 e^{-b_1 n},\ n \geqslant 1,$$

where

$$A_2^n \triangleq \{\| \bar{Q}(t) \| > 0, t \in [0, \theta(n(T-\epsilon_2))],\ \bar{Q}_{12}(t) > 0,\ t \in [\theta(n\epsilon_1), \theta(n(T-\epsilon_2))],$$
$$\theta(n(T-\epsilon_2)) - \theta(n\epsilon_1) = n(T-\epsilon_1-\epsilon_2)\}$$

Now by Cramér's classical theorem we obtain that for any $\epsilon_3 > 0$ the conditional probability is

$$\mathbf{P}(A_3^n \mid A_2^n) \geqslant 1 - a_2 e^{-b_2 n},$$

where

$$A_3^n \triangleq \{Q_{22}(\theta(n(T-\epsilon_1))) > n(T-\epsilon_1-\epsilon_2) - n\epsilon_3\}.$$

Clearly, $A_2^n \cap A_3^n \subset B$ if we take $c = T - \epsilon_1 - \epsilon_2 - \epsilon_3$. Thus,

$$\mathbf{P}(B) \geqslant \mathbf{P}(A_2^n \cap A_3^n) \geqslant 1 - a e^{-bn},\ n \geqslant 1,$$

where $a = a_1 + a_2$, $b = \min\{b_1, b_2\}$. Q.E.D.

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